

CRC HANDBOOK OF

LIE GROUP  
ANALYSIS OF  
DIFFERENTIAL  
EQUATIONS

VOLUME 1

SYMMETRIES  
EXACT SOLUTIONS  
AND  
CONSERVATION LAWS

EDITED BY

N. H. IBRAGIMOV

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**N. H. IBRAGIMOV**

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## *A. Apparatus of Group Analysis*

# I

## Lie Theory of Differential Equations

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Lie, in his group theoretic investigations of differential equations, found it necessary to distinguish clearly between two approaches: One is natural and deals with the totality of solutions of a given differential equation. The other is based on regarding the differential equation as a surface in the space of independent and dependent variables together with the derivatives involved in the given equation. To illustrate this, consider a first-order ordinary differential equation [ODE],

$$F(x, y, y') = 0. \quad (i)$$

In the first approach, the symmetry group of Equation i is regarded as a one-parameter group (with parameter  $a$ ) of point transformations of the  $(x, y)$  plane,

$$\bar{x} = \varphi(x, y, a), \quad \bar{y} = \psi(x, y, a), \quad (ii)$$

such that any solution  $h(x)$  of Equation i (i.e.,  $F(x, h(x), h'(x)) = 0$ , identically in  $x$  from some interval) is converted into a solution of Equation i in the following way. Consider the integral curve

$$(x, y = h(x)). \quad (iii)$$

Fix the parameter  $a$  in Equation ii and apply the Transformation ii to the integral curve iii. This yields a curve given by

$$(\bar{x} = \varphi(x, h(x), a), \bar{y} = \psi(x, h(x), a)), \quad (iv)$$

which, according to the first approach, is an integral curve. This integral

curve, after elimination of  $x$  from Equations iv, can be rewritten in the form

$$\bar{y} = H(\bar{x}, a). \quad (\text{v})$$

Because  $\bar{x}$  is arbitrary we can again denote it by  $x$ . Hence, the original solution  $h(x)$  of Equation i is converted by the symmetry group into a one-parameter family of solutions  $H(x, a)$  of Equation i.

In the second approach, the differential equation is considered as a surface in the three-dimensional space of variables  $x, y, p$  given by

$$F(x, y, p) = 0. \quad (\text{vi})$$

Here  $x, y$ , and  $p$  are considered to be three independent variables that transform as

$$\bar{x} = \varphi(x, y, a), \quad \bar{y} = \psi(x, y, a), \quad \bar{p} = D\psi/D\varphi, \quad (\text{vii})$$

where  $D = \partial/\partial x + p \partial/\partial y$ . This is, in fact, a one-parameter group and is called the *first prolongation* of the group of point transformations. A symmetry group, in the sense of this second approach, is defined as a group of transformations such that its first prolongation leaves invariant the surface given by Equation vi. The constraint on the transformation law for  $p$  that appears in Equation vii provides a connection with the first approach because the prolongation is consistent with the transformation law for first derivatives with the identification  $p = y'$ . Equally important is the fact that this constraint provides an algorithm for finding symmetry groups.

It is clear from the second approach that the symmetry group of Equation i is identical to the invariance group for the surface given by Equation vi and does not depend on the existence of solutions of the differential equation. Because of this fundamental role played by the surface given by Equation vi, it is called the *frame* of the differential equation.

In integrating differential equations, a decisive step is that of simplifying the frame. For such a purpose, it is sufficient to "straighten out" the one-parameter symmetry group i.e., to reduce its action to a translation by a suitable change of the variables  $x$  and  $y$ . This automatically simplifies the equation by converting its frame into a cylinder, i.e., the explicit dependence on one of the variables  $x$  or  $y$  has been eliminated.

For illustration, we give the following example.

**Example.** The Riccati equation,  $y' + y^2 - 2/x^2 = 0$ , admits the group  $G$  of point transformations  $\bar{x} = xe^a$  and  $\bar{y} = ye^{-a}$  because the frame of this equation,  $p + y^2 - 2/x^2 = 0$ , is invariant under the dilations in the space of three variables  $(x, y, p)$   $\bar{x} = xe^a$ ,  $\bar{y} = ye^{-a}$ ,  $\bar{p} = pe^{-2a}$ . These dilations are obtained by the prolongation of  $G$ . The group is straightened out by the

change of variables  $t = \ln x$ ,  $u = xy$ . In these variables the transformations of the group  $G$  are written  $\bar{t} = t + a$ ,  $\bar{u} = u$ . Its prolongation to the frame space  $(t, u, q)$  is given by  $\bar{t} = t + a$ ,  $\bar{u} = u$ ,  $\bar{q} = q$ , which is obtained by setting  $p = q$  in Formula vii. In the new variables the frame of the Riccati equation becomes the parabolic cylinder  $q + u^2 - u - 2 = 0$ . As a result we have transformed the original Riccati equation into the following integrable one:  $u' + u^2 - u - 2 = 0$ .

Central to this book is the remarkable discovery by Lie that the group approach provides a unified explanation for the seemingly disparate (diverse and ad hoc) integration methods used to solve ordinary differential equations. Moreover, Lie [1883], [1884] gave a group classification of all arbitrary-order ordinary differential equations. In this way he identified all equations that can be reduced to lower-order equations or completely integrated by the application of group theoretic methods.

The purpose of Part I is as follows: (1) to systematize the relevant results of Lie; (2) to guide the reader through the totality of Lie group methods with a minimum number of theoretical constructions; (3) to assist the reader in developing skills in applying these results and methods.

# One-Parameter Transformation Groups

## 1.1. LOCAL ONE-PARAMETER POINT TRANSFORMATION GROUPS

We shall consider invertible transformations of the  $(x, y)$  plane

$$\bar{x} = \varphi(x, y, a), \quad \bar{y} = \psi(x, y, a), \quad (1.1)$$

depending upon a real parameter  $a$ , where we impose the conditions

$$\varphi|_{a=0} = x, \quad \psi|_{a=0} = y. \quad (1.2)$$

We say that these transformations form a *one-parameter group*  $G$  if the successive action of two transformations is equivalent to the action of another transformation of the form 1.1. This group property can always be recast, by a suitable choice of parameter, as follows:

$$\begin{aligned} \varphi(\bar{x}, \bar{y}, b) &= \varphi(x, y, a + b), \\ \psi(\bar{x}, \bar{y}, b) &= \psi(x, y, a + b). \end{aligned} \quad (1.3)$$

In practice, it often happens that the group property is valid only locally, i.e., only for  $a$  and  $b$  sufficiently small. In this case  $G$  is referred to as a *local one-parameter transformation group*. If the group property is valid for all  $a$  and  $b$  from some fixed interval,  $G$  is referred to as a *global group*. It is local groups that are used in group analysis. For brevity, we simply call them groups.

Transformations 1.1 are called *point transformations* (unlike contact transformations, where the transformed values also depend on the derivative  $y'$ ), and the group  $G$  is called a group of point transformations. It is readily seen

from Formulas 1.2 and 1.3 that the inverse transformation can be obtained by changing the sign of the group parameter:

$$x = \varphi(\bar{x}, \bar{y}, -a), \quad y = \psi(\bar{x}, \bar{y}, -a). \quad (1.4)$$

Let  $T_a$  denote the transformation 1.1 of a point  $(x, y)$  into a point  $(\bar{x}, \bar{y})$ ,  $I$  the identity transformation,  $T_a^{-1}$  the transformation inverse to  $T_a$ , and  $T_b T_a$  the composition of two transformations. Then one may summarize properties 1.2–1.4 as follows.

**Definition 1.1.** The set  $G$  of transformations  $T_a$  is a local one-parameter group if the following hold:

1.  $T_0 = I \in G$
2.  $T_b T_a = T_{a+b} \in G$
3.  $T_a^{-1} = T_{-a} \in G$

where  $a$  and  $b$  are sufficiently small.

We shall represent the functions  $\varphi$  and  $\psi$  via their Taylor series expansions with respect to the parameter  $a$  in the neighborhood of  $a = 0$  and write the infinitesimal Transformation 1.1 as follows:

$$\bar{x} \approx x + \xi(x, y)a, \quad \bar{y} \approx y + \eta(x, y)a, \quad (1.1')$$

where

$$\xi(x, y) = \left. \frac{\partial \varphi(x, y, a)}{\partial a} \right|_{a=0}, \quad \eta(x, y) = \left. \frac{\partial \psi(x, y, a)}{\partial a} \right|_{a=0}. \quad (1.5)$$

For example, in the case of rotations

$$\bar{x} = x \cos a + y \sin a, \quad \bar{y} = y \cos a - x \sin a,$$

the infinitesimal transformation is given by

$$\bar{x} \approx x + ya, \quad \bar{y} \approx y - xa.$$

The vector  $(\xi, \eta)$  given by Formula 1.5 is a tangent vector (at the point  $(x, y)$ ) to the curve determined by the totality of transformed points  $(\bar{x}, \bar{y})$ . That is why it is called a tangent vector field of the group  $G$ .

Given an infinitesimal transformation 1.1', a one-parameter group can be completely determined by the following Lie equations with appropriate initial

conditions:

$$\begin{aligned}\frac{d\varphi}{da} &= \xi(\varphi, \psi), & \varphi|_{a=0} &= x, \\ \frac{d\psi}{da} &= \eta(\varphi, \psi), & \psi|_{a=0} &= y.\end{aligned}\tag{1.6}$$

A tangent vector field can be written in terms of the first-order differential operator

$$X = \xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y}.\tag{1.7}$$

Unlike the vector  $(\xi, \eta)$ ,  $X$  transforms as a scalar under a change of variables. Lie called the operator 1.7 a *symbol* of the infinitesimal Transformation 1.1'. The terms *infinitesimal operator*, *group operator*, *Lie operator*, and *group generator* came into use later. All these terms will be used interchangeably.

**Definition 1.2.** A function  $F(x, y)$  is an invariant of the group of transformations 1.1 if for each point  $(x, y)$  it is constant along the trajectory determined by the totality of transformed points  $(\bar{x}, \bar{y})$ :

$$F(\bar{x}, \bar{y}) = F(x, y).$$

**Theorem 1.1.** The function  $F(x, y)$  is an invariant of the group  $G$  with the symbol  $X$  given by Formula 1.7 iff it satisfies the partial differential equation

$$XF \equiv \xi(x, y) \frac{\partial F}{\partial x} + \eta(x, y) \frac{\partial F}{\partial y} = 0.\tag{1.8}$$

Hence any one-parameter group of point transformations of the plane has only one independent invariant. One can take this invariant to be the left-hand side of a first integral  $J(x, y) = C$  of the characteristic equation

$$\frac{dx}{\xi(x, y)} = \frac{dy}{\eta(x, y)}.\tag{1.8'}$$

Then any other invariant is a function of  $J$ .

The concepts just stated can be readily generalized to the multidimensional case, where instead of transformation groups of the plane one considers transformations

$$\bar{x}^i = f^i(x, a), \quad i = 1, \dots, n,\tag{1.9}$$

of the  $n$ -dimensional space of points  $x = (x^1, \dots, x^n)$ . Let us focus on this multidimensional case and consider the system of equations

$$F_1(x) = 0, \dots, F_s(x) = 0, \quad s < n. \quad (1.10)$$

We suppose that the rank of the matrix  $\|\partial F_k / \partial x^1\|$  equals  $s$  at every point  $x$  that satisfies System 1.10. Then System 1.10 determines an  $(n - s)$ -dimensional surface  $M$ .

**Definition 1.3.** System 1.10 is invariant under the transformation group  $G$  (or admits  $G$ ) if any point  $x$  of the surface  $M$  moves along this surface under the action of  $G$ , i.e.,  $\bar{x} \in M$  if  $x \in M$ .

**Theorem 1.2.** The system of Equations 1.10 is invariant under the group  $G$  of Transformations 1.9 with the symbol

$$X = \xi^i(x) \frac{\partial}{\partial x^i}, \quad \xi^i(x) = \left. \frac{\partial f^i(x, a)}{\partial a} \right|_{a=0}, \quad (1.11)$$

iff

$$XF_k|_M = 0, \quad k = 1, \dots, s, \quad (1.12)$$

where the notation  $|_M$  means evaluated on  $M$ .

This theorem is useful for finding the group admitted by a given system of equations. When the functions  $F_k(x)$  are known, the coordinates  $\xi^i(x)$  of Operator 1.11 are determined from Equations 1.12. Then one may obtain Transformations 1.9 of the group admitted by solving the Lie equations

$$\frac{d\bar{x}^i}{da} = \xi^i(\bar{x}), \quad \bar{x}^i|_{a=0} = x^i, \quad i = 1, \dots, n.$$

Conversely, in order to find an invariant system of equations for a given group  $G$ , it is convenient to use the following theorem on the representation of invariant equations via group invariants. Each one-parameter group  $G$  of Transformations 1.9 has exactly  $n - 1$  functionally independent invariants. Any set of  $n - 1$  functionally independent invariants is called a *basis of invariants* for  $G$ . A basis of invariants for a group  $G$  with symbol 1.11 can, in principle, be constructed by solving the characteristic system

$$\frac{dx^1}{\xi^1} = \dots = \frac{dx^n}{\xi^n} \quad (1.11')$$

**Theorem 1.3.** Let the system of equations 1.10 admit the group  $G$ , and assume its tangent vector  $\xi(x)$  is not equal to zero on the surface  $M$  determined by Equations 1.10. Then there exist functions  $\Phi$  such that one may rewrite System 1.10 equivalently as follows:

$$\Phi_k(J_1(x), \dots, J_{n-1}(x)) = 0, \quad k = 1, \dots, s, \quad (1.10')$$

where  $J_1(x), \dots, J_{n-1}(x)$  is a basis of invariants of the group  $G$  (i.e., a set of all functionally independent invariants). Equations 1.10 and 1.10' are equivalent in the sense that they determine the same surface  $M$ .

**Example.** Consider the paraboloid given by

$$x^2 + y^2 - z = 0$$

with the origin excluded. This paraboloid is equivalently given by

$$\frac{x^2 + y^2}{z} - 1 = 0.$$

The group  $G$  of inhomogeneous dilations  $\bar{x} = xe^a$ ,  $\bar{y} = ye^a$ , and  $\bar{z} = ze^{2a}$  moves points along this parabolic surface, hence both equations describing the paraboloid are invariant under  $G$ . However, it can be easily verified that the function  $F(x, y, z) = x^2 + y^2 - z$  under the action of  $G$  becomes  $F(\bar{x}, \bar{y}, \bar{z}) = e^{2a}F(x, y, z)$ . Hence  $F$  is not invariant under  $G$ , whereas the function  $\Phi(x, y, z) = (x^2 + y^2)/z - 1$  is invariant under  $G$ .

In integrating ordinary differential equations we shall use the following simple theorem on what is called similarity of one-parameter groups. Here, we formulate this theorem only in the case of transformation groups on the plane.

**Theorem 1.4.** Any one-parameter group  $G$  of Transformations 1.1 can be reduced under a suitable change of variables

$$t = t(x, y), \quad u = u(x, y)$$

to the translation group  $\bar{t} = t + a$  and  $\bar{u} = u$  with the symbol  $X = \partial/\partial t$ . Such variables  $t$  and  $u$  are referred to as *canonical variables*.

This can be seen by the following argument. By a change of variables, the differential operator given by Formula 1.7 is transformed according to the formula

$$X = X(t) \frac{\partial}{\partial t} + X(u) \frac{\partial}{\partial u}, \quad (1.13)$$

where  $X(t)$  and  $X(u)$  denote the action of the differential operator  $X$  on the functions  $t(x, y)$  and  $u(x, y)$ , respectively. In order that Operator 1.13 becomes  $X = \partial/\partial t$ , the equations

$$X(t) = 1 \quad \text{and} \quad X(u) = 0 \quad (1.14)$$

must be satisfied.

For example, for the dilation group  $\bar{x} = xe$  and  $\bar{y} = ye^{2a}$  with the operator  $X = x \partial/\partial x + 2y \partial/\partial y$ , Equations 1.14 are readily solved and yield  $t = \ln x + g(yx^{-2})$  and  $u = h(yx^{-2})$ , where  $g$  and  $h$  are arbitrary functions. One has the freedom to choose the functions  $g$  and  $h$ . We take  $g = 0$  and  $h = yx^{-2}$ , i.e., the change of variables  $t = \ln x$ ,  $u = y/x^2$ . This reduces the dilation group to the group of translations

$$\bar{t} = \ln \bar{x} = \ln x + a = t + a, \quad \bar{u} = \frac{\bar{y}}{\bar{x}^2} = \frac{y}{x^2} = u.$$

## 1.2. PROLONGATION FORMULAS

Now we write the transformation formulas for the derivatives  $y', y''$  corresponding to the point transformations. It is convenient to use the operator of total differentiation

$$D = \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + y'' \frac{\partial}{\partial y'} + \cdots,$$

where  $D$  is used to distinguish this operator from the operator of partial differentiation  $\partial/\partial x$ . Then the derivative transformations are given by

$$\bar{y}' \equiv \frac{d\bar{y}}{d\bar{x}} = \frac{D\psi}{D\varphi} = \frac{\psi_x + y'\psi_y}{\varphi_x + y'\varphi_y} \equiv P(x, y, y', a), \quad (1.15)$$

$$\bar{y}'' \equiv \frac{d\bar{y}'}{d\bar{x}} = \frac{DP}{D\varphi} = \frac{P_x + y'P_y + y''P_{y'}}{\varphi_x + y'\varphi_y}. \quad (1.16)$$

If we start from a group  $G$  of transformations 1.1 and add Formula 1.15, we obtain the first prolongation  $G_1$  acting on the space of three variables  $(x, y, y')$ ; after adding Formula 1.16, we obtain the second prolongation  $G_2$  acting on the space  $(x, y, y', y'')$ .

Substituting the infinitesimal transformation  $\bar{x} = x + a\xi$  and  $\bar{y} = y + a\eta$  into Formulas 1.15 and 1.16 and neglecting the terms up to  $o(a)$ , we obtain

the infinitesimal transformations of the derivatives

$$\begin{aligned}\bar{y}' &= \frac{y' + aD(\eta)}{1 + aD(\xi)} \approx [y' + aD(\eta)][1 - aD(\xi)] \\ &\approx y' + [D(\eta) - y'D(\xi)]a \equiv y' + a\zeta_1, \\ \bar{y}'' &= \frac{y'' + aD(\zeta_1)}{1 + aD(\xi)} \approx [y'' + aD(\zeta_1)][1 - aD(\xi)] \\ &\approx y'' + [D(\zeta_1) - y''D(\xi)]a \approx y'' + a\zeta_2.\end{aligned}$$

Hence the symbols of the groups  $G_1$  and  $G_2$  are equal to

$$X_1 = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \zeta_1 \frac{\partial}{\partial y'}, \quad \zeta_1 = D(\eta) - y'D(\xi), \quad (1.17)$$

$$X_2 = X_1 + \zeta_2 \frac{\partial}{\partial y''}, \quad \zeta_2 = D(\zeta_1) - y''D(\xi). \quad (1.18)$$

They are referred to as the first- and the second-order prolongations of Operator 1.7. Sometimes we shall also call the expressions for the additional coordinates  $\zeta_1$  and  $\zeta_2$ ,

$$\zeta_1 = D(\eta) - y'D(\xi) = \eta_x + (\eta_y - \xi_x)y' - y'^2\xi_y, \quad (1.17')$$

$$\begin{aligned}\zeta_2 = D(\zeta_1) - y''D(\xi) &= \eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 \\ &\quad - y'^3\xi_{yy} + (\eta_y - 2\xi_x - 3y'\xi_y)y'',\end{aligned} \quad (1.18')$$

prolongation formulas.

In dealing with the multidimensional situation (with independent variables  $x^i$ ,  $i = 1, \dots, n$ , and dependent variables  $u^\alpha$ ,  $\alpha = 1, \dots, m$ , instead of  $x$  and  $y$ ) one may recast Prolongations 1.17 and 1.18 of the operator

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta \frac{\partial}{\partial u^\alpha}$$

as follows:

$$X_1 = X + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha}, \quad X_2 = X_1 + \zeta_{ij}^\alpha \frac{\partial}{\partial u_{ij}^\alpha},$$

where

$$\zeta_i^\alpha = D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j), \quad (1.17'')$$

$$\zeta_{ij}^\alpha = D_j(\zeta_i^\alpha) - u_{ik}^\alpha D_j(\xi^k), \quad (1.18'')$$

and

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots.$$

The convention of summation over repeated indices is adopted here.

### 1.3. GROUPS ADMITTED BY DIFFERENTIAL EQUATIONS

Let  $G$  be the group of point transformations and let  $G_1$  and  $G_2$  be the first and the second prolongations defined in Section 1.2.

**Definition 1.4.** A first-order ordinary differential equation

$$F(x, y, y') = 0$$

admits the group  $G$  if the frame of the given equation, i.e., a two-dimensional surface  $F(x, y, p) = 0$  in the three-dimensional space of variables  $x$ ,  $y$ , and  $p = y'$  is invariant (see Definition 1.3) under the first prolongation  $G_1$  of  $G$ . Analogously a second-order differential equation

$$F(x, y, y', y'') = 0 \tag{1.19}$$

admits a group  $G$  if its frame (a three-dimensional surface in the space  $x, y, y', y''$ ) is invariant under the second prolongation  $G_2$ .

Evidently this definition can be generalized to higher-order differential equations as well as to systems of partial differential equations.

The terms “groups admitted by differential equations” and “symmetry groups” are used interchangeably in the literature.

Under the action of a group admitted by a differential equation, any solution of the equation is converted into a solution of the same equation. Often in the literature one finds the converse statement taken as the definition for symmetry groups, namely:

**Definition 1.4'.** A symmetry group of a differential equation is a group that converts every solution of the equation under consideration into a solution of the same equation.

Under conditions of solvability, Definitions 1.4 and 1.4' are equivalent (see Lie [1891], Chapter 6, Section 1; for a recent treatment of locally solvable systems see Olver [1986], Section 2.6). However, because of the work of

H. Lewy [1957], we know there are exceptional cases of equations without solution. In these cases the two definitions are not equivalent.

The construction of differential equations admitting a given group can be readily realized with the help of Theorem 1.3 on the representation of invariant equations in terms of group invariants.

If we fix our attention on constructing the most general form of second-order ODEs that admit a given group with generator  $X$ , then one must find a basis of invariants for  $X$ , i.e., solve

$$XJ(x, y, y', y'') = 0.$$

The theory of characteristics says there are three functionally independent invariants. It is convenient to find these invariants successively by solving the sequence of equations

$$XJ(x, y) = 0, \quad XJ(x, y, y') = 0, \quad XJ(x, y, y', y'') = 0. \quad (1.20)$$

The first equation has one functionally independent solution, which we denote by  $u(x, y)$ . The second equation has two functionally independent solutions; we select one of them to be  $u(x, y)$  (because  $Xu = 0$ ) and we denote the second one by  $v(x, y, y')$ . Note that  $v(x, y, y')$  must depend on  $y'$ , otherwise the first of Equations 1.20 would have two functionally independent invariants. The solution  $v(x, y, y')$  is called a first-order differential invariant of the group  $G$ . Similarly, we find from the third of Equations 1.20 exactly one additional functionally independent (of  $u$  and  $v$ ) invariant  $w(x, y, y', y'')$ , which is called a second-order differential invariant of the group  $G$ . The choice of invariants is unique up to functional dependence. If one looks for the most general form of first-order ODEs then it is only necessary to consider the first two equations from Equations 1.20.

**Example.** Let  $G$  be the group of Galilean transformations  $\bar{x} = x + ay$  and  $\bar{y} = y$  with generator  $X = y \partial / \partial x$ . This group has the invariant  $u = y$ . The first and second prolongations of the operator  $X$  are easily calculated by Formulas 1.17 and 1.18 and turn out to be

$$X_1 = y \frac{\partial}{\partial x} - y'^2 \frac{\partial}{\partial y'}, \quad X_2 = y \frac{\partial}{\partial x} - y'^2 \frac{\partial}{\partial y'} - 3y'y'' \frac{\partial}{\partial y''}.$$

From the characteristic equations

$$\frac{dx}{y} = -\frac{dy'}{y'^2} = -\frac{dy''}{3y'y''}$$

we obtain the first- and second-order differential invariants:

$$v = \frac{y}{y'} - x, \quad w = \frac{y''}{y'^3}.$$

In accordance with Theorem 1.3 the most general invariant equation (except singular ones) for prolongations  $G$  and  $G$  can be written as  $v = F(u)$  and  $w = F(u, v)$ , respectively. Substituting here the expressions for the invariants  $u$ ,  $v$ , and  $w$ , we obtain the following general first- and second-order differential equations admitting the Galilean group:

$$y' = \frac{y}{x + F(y)}, \quad y'' = y'^3 F\left(y, \frac{1}{y'} - \frac{x}{y}\right).$$

In more complex cases the following theorem (Lie [1891], Chapter 16, Section 5) simplifies calculation of the second- and higher-order differential invariants.

**Theorem 1.5.** Let the invariant  $u(x, y)$  and the first-order differential invariant  $v(x, y, y')$  be known for a given group  $G$ . Then the derivative

$$w = \frac{dv}{du} = \frac{v_x + y'v_y + y''v_{y'}}{u_x + y'u_y} \equiv \frac{Dv}{Du}$$

is a second-order differential invariant. As a result of subsequent differentiation one may obtain the higher-order differential invariants  $d^2v/du^2$ ,  $d^3v/du^3, \dots$ .

Several differential equations of the first and second orders are listed in Sections 8.1 and 8.3 along with the operators they admit. These results are obtained with the help of Theorems 1.3 and 1.5.

## 1.4. INTEGRATION AND REDUCTION OF ORDER

Group theory elucidates many methods and results on the integration of equations that are widely used in practice. This permits one to understand connections between different methods and to unify them. Here we discuss the simplest applications of one-parameter groups to the problems of integration and reduction of order for ordinary differential equations.

For the integration of first-order equations, two group theoretical methods are presented. The first provides a method for finding an integrating factor (Section 1.4.1). The second, the method of canonical variables (Section 1.4.2), provides a method for finding a suitable change of variables. This last

method, in the case of higher order equations, becomes a method for reducing the order of the equation.

For these and more general group theoretic methods of integration of ODEs see the exhaustive presentation by Lie [1891]. Helpful presentations can also be found in Bianchi [1918] and Dickson [1924] as well as in more recent books by Olver [1986], Bluman and Kumei [1989] Ibragimov [1989a], [1991], and Stephani [1989].

### 1.4.1. INTEGRATING FACTOR

This method is applicable to first-order equations only. Consider a first-order ordinary differential equation written in the symmetric form

$$Q(x, y) dx - P(x, y) dy = 0. \quad (1.21)$$

It is equivalent to the following partial differential equation:

$$P \frac{\partial F}{\partial x} + Q \frac{\partial F}{\partial y} = 0. \quad (1.22)$$

The left-hand side of any integral  $F(x, y) = \text{const.}$  of Equation 1.21 is a solution of Equation 1.22 and, conversely, any solution  $F(x, y)$  of Equation 1.22 equated to an arbitrary constant determines an integral of Equation 1.21.

**Theorem 1.6** (Lie [1875]). Equation 1.21,

$$Q(x, y) dx - P(x, y) dy = 0,$$

admits a one-parameter group with generator 1.7,

$$X = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}$$

iff the function

$$\mu = \frac{1}{\xi Q - \eta P} \quad (1.23)$$

is an integrating factor for Equation 1.21.

**First Example.** Consider again the Riccati equation

$$y' + y^2 = \frac{2}{x^2}. \quad (1.24)$$

This equation admits the one-parameter group of dilations  $\bar{x} = xe^a$  and  $\bar{y} = ye^{-a}$  with the generator

$$X = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}. \quad (1.25)$$

Rewriting Equation 1.24 as

$$dy + \left( y^2 - \frac{2}{x^2} \right) dx = 0 \quad (1.24')$$

and using Formula 1.23, we obtain the integrating factor

$$\mu = \frac{x}{x^2 y^2 - xy - 2}.$$

After multiplying by this factor, Equation 1.24' becomes

$$\frac{x dy + (xy^2 - 2/x) dx}{x^2 y^2 - xy - 2} = d \left( \ln x + \frac{1}{3} \ln \frac{xy - 2}{xy + 1} \right) = 0.$$

Thus we find the general solution of Equation 1.24 in the form

$$y = \frac{2x^3 + C}{x(x^3 - C)}.$$

**Second Example—A Group Theoretical Basis for the Method of Variation of Parameters.** Let us apply Lie's formula to integrate the inhomogeneous linear equation

$$y' + R(x)y = Q(x). \quad (1.26)$$

In this case a symmetry group is given by the superposition principle. Namely, if  $z_0(x) = \exp[\int R(x) dx]$  is a particular solution of the homogeneous equation

$$z' + R(x)z = 0, \quad (1.26')$$

then the one-parameter family of transformations

$$\bar{x} = x, \quad \bar{y} = y + az_0(x)$$

converts any solution of Equation 1.26 into a solution of the same equation.

These transformations form a one-parameter group  $G$  with the symbol

$$X = z_0(x) \frac{\partial}{\partial y}.$$

Here, one can use Definition 1.4' and conclude that  $G$  is a symmetry group for Equation 1.26. Now we rewrite Equation 1.26 in the form

$$(Q - Ry) dx - dy = 0,$$

and, using Formula 1.23, we find the integrating factor  $\mu = -1/z_0(x)$ , or

$$\mu = -e^{\int R(x) dx}.$$

Therefore

$$[dy + (Ry - Q) dx] e^{\int R(x) dx} = dF,$$

for some function  $F$ . It follows that

$$\frac{\partial F}{\partial y} = e^{\int R(x) dx}, \quad \frac{\partial F}{\partial x} = R(x)y e^{\int R(x) dx} - Q(x)e^{\int R(x) dx}.$$

By integrating the first equation one may obtain the expression  $F = y \exp[\int R(x) dx] + f(x)$ . The substitution of this expression into the second equation yields

$$f'(x) = -Q(x)e^{\int R(x) dx}, \quad \text{or} \quad f(x) = -\int Q(x)e^{\int R(x) dx} dx.$$

Substituting this back into the expression for  $F(x, y)$  and setting  $F = C$ , we find

$$ye^{\int R(x) dx} - \int Q(x)e^{\int R(x) dx} dx = C.$$

By solving this equation with respect to  $y$  one obtains the general solution of Equation 1.26:

$$y = e^{-\int R dx} \left( \int Q e^{\int R dx} dx + C \right).$$

#### 1.4.2. METHOD OF CANONICAL VARIABLES

First we note that if a differential equation admits a group  $G$  in one coordinate system, it admits the group in any other coordinate system. Therefore, we simplify the group by choosing canonical variables  $t$  and  $u$  according to Theorem 1.4 and reduce the action of  $G$  to translations in one of these variables, say,  $t$ . Then our equation written in terms of  $t$  and  $u$  will be invariant with respect to translations in  $t$ . This means that the trans-

formed equation does not depend explicitly on  $t$ . Therefore, in canonical variables, we can integrate this equation by quadratures if it is first-order, or if it is of higher-order, reduce the order by 1.

**First Example.** Consider equation 1.24. For the group with generator 1.25, the canonical variables are

$$t = \ln x \quad \text{and} \quad u = xy.$$

In these variables Equation 1.24 is written as

$$u' + u^2 - u - 2 = 0. \quad (1.24'')$$

This equation is readily integrated and yields

$$\ln \frac{u+1}{u-2} - 3t = \text{const.}$$

After substituting the expressions for  $t$  and  $u$  in terms of  $x$  and  $y$ , we reproduce the solution given in Section 1.4.1.

**Second Example.** The second-order linear equation

$$y'' + f(x)y = 0 \quad (1.27)$$

admits the group of dilations in  $y$  with the generator

$$X = y \frac{\partial}{\partial y}. \quad (1.28)$$

Here the canonical variables are  $u = x$  and  $t = \ln y$ . In these variables Equation 1.27 becomes

$$u'' - u' + f(u)u'^3 = 0.$$

It does not depend explicitly on  $t$ . That is why its order can be reduced by the standard substitution  $u' = p(u)$  and the problem becomes one of integrating the Riccati equation

$$\frac{dp}{du} + f(u)p^2 - 1 = 0.$$

The method of canonical variables gives a second group theoretic basis for the method of variation of parameters (see, e.g., Ibragimov [1989a]). In this way, in contrast to the integrating-factor method, one is provided with a group theoretic approach to the method of variation of parameters for higher-order equations.

### 1.4.3. INVARIANT DIFFERENTIATION

If a second- or higher-order equation admits a one-parameter group, one can reduce the order of the equation with the help of Theorems 1.3 and 1.5.

Here we focus on second-order ODEs admitting a one-parameter group  $G$ . Denote by  $u$  an invariant of  $G$  and by  $v$  its first-order differential invariant. According to Theorem 1.5 a second-order differential invariant can be chosen in the form  $w = dv/du$ . Then by Theorem 1.3 the differential equation can be rewritten as

$$\frac{dv}{du} = F(u, v). \quad (1.29)$$

By this we have reduced the order. If we find an integral of Equation 1.29 (e.g., of the original second-order equation) in the sense

$$\Phi(u, v, C) = 0 \quad (1.30)$$

then the solution of our second-order equation automatically reduces to quadratures. Indeed, the substitution of the known expressions for invariants  $u(x, y)$  and  $v(x, y, y')$  into Equation 1.30 results in a first-order differential equation that admits the group  $G$ . This is the consequence of the fact that  $u$  and  $v$  are invariants.

**Example.** Let us reduce the order of Equation 1.27 by the method of invariant differentiation. The first prolongation of operator 1.28 is

$$X_1 = y \frac{\partial}{\partial y} + y' \frac{\partial}{\partial y'}.$$

Its invariants are  $u = x$  and  $v = y'/y$ . In accordance with theorem 1.5, we calculate the second-order differential invariant  $w = dv/du$ :

$$\frac{dv}{du} = \frac{y''}{y} - \frac{y'^2}{y^2} = \frac{y''}{y} - v^2.$$

It follows that  $y''/y = dv/du + v^2$ . The substitution of this expression into Equation 1.27 gives Equation 1.29 in the form of the following Riccati equation:

$$\frac{dv}{du} + u^2 + f(u) = 0.$$

## 1.5. DETERMINING EQUATION

Now we consider the problem of constructing the group admitted by a given second-order Equation 1.19. In accordance with Definition 1.4 and Theorem 1.2, the infinitesimal criterion for invariance is

$$X^{(2)}F|_{F=0} \equiv (\xi F_x + \eta F_y + \zeta_1 F_{y'} + \zeta_2 F_{y''})|_{F=0} = 0 \quad (1.31)$$

where  $\zeta_1$  and  $\zeta_2$  are calculated by the prolongation formulas (Equations 1.17' and 1.18'). Equation 1.31 is referred to as the determining equation for the group admitted by Equation 1.19.

Furthermore, we shall treat differential equations written in the form

$$y'' = f(x, y, y'). \quad (1.32)$$

In this case the determining equation, becomes

$$\begin{aligned} &\eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 - y'^3\xi_{yy} \\ &\quad + (\eta_y - 2\xi_x - 3y'\xi_y)f - [\eta_x + (\eta_y - \xi_x)y' - y'^2\xi_y]f_{y'} \\ &\quad - \xi f_x - \eta f_y = 0. \end{aligned} \quad (1.33)$$

Here  $f(x, y, y')$  is a known function and the coordinates  $\xi$  and  $\eta$  are unknown functions of  $x$  and  $y$ . Because the left-hand side of Equation 1.33 includes (besides  $x$  and  $y$ )  $y'$  considered as an independent variable, the determining equation can be split into several independent equations. As a result we obtain an overdetermined system of differential equations for  $\xi$  and  $\eta$ . Solving this system of determining equations, we find all operators admitted by Equation 1.32.

**First Example.** Let us find all operators

$$X = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y},$$

admitted by

$$y'' + \frac{1}{x}y' - e^y = 0. \quad (1.34)$$

Here  $f = e^y - (1/x)y'$  and Equation 1.33 becomes

$$\begin{aligned} &\eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 - y'^3\xi_{yy} \\ &\quad + (\eta_y - 2\xi_x - 3y'\xi_y)\left(e^y - \frac{y'}{x}\right) + \frac{1}{x}[\eta_x + (\eta_y - \xi_x)y' - y'^2\xi_y] \\ &\quad - \xi \frac{y'}{x^2} - \eta e^y = 0. \end{aligned}$$

The left-hand side of this equation is a polynomial of third degree in  $y'$ . That is why the determining equation is split into the following four equations:

$$\begin{aligned}
 (y')^3: \quad \xi_{yy} &= 0; \\
 (y')^2: \quad \eta_{yy} - 2\xi_{xy} + \frac{2}{x}\xi_y &= 0; \\
 y': \quad 2\eta_{xy} - \xi_{xx} + \left(\frac{\xi}{x}\right)_x - 3\xi_y e^y &= 0; \\
 (y')^0: \quad \eta_{xx} + \frac{1}{x}\eta_x + (\eta_y - 2\xi_x - \eta)e^y &= 0.
 \end{aligned} \tag{1.35}$$

It follows from the first two equations that

$$\xi = p(x)y + a(x) \quad \text{and} \quad \eta = \left(p' + \frac{p}{x}\right)y^2 + \left[2\left(a' - \frac{a}{x}\right) + q(x)\right]y + b(x).$$

We substitute these expressions for  $\xi$  and  $\eta$  into the last two of Equations 1.35. Because  $\xi$  and  $\eta$  depend on  $y$  polynomially and the left-hand sides of these last two equations contain  $e^y$ , it follows that

$$\xi_y = 0 \quad \text{and} \quad \eta_y - 2\xi_x - \eta = 0.$$

The solution of these equations is given by

$$\xi = a(x) \quad \text{and} \quad \eta = -2a'(x).$$

After this the third equation of Equations 1.35 becomes

$$\left(a' - \frac{a}{x}\right)' = 0,$$

which gives  $a = C_1 x \ln x + C_2 x$ . The last equation of Equations 1.35 is valid identically.

As a result we obtain the general solution of Equations 1.35:

$$\xi = C_1 x \ln x + C_2 x \quad \text{and} \quad \eta = -2[C_1(1 + \ln x) + C_2],$$

with constant coefficients  $C_1$  and  $C_2$ . By virtue of the linearity of the determining equations the general solution can be represented as a linear combination of the following two independent solutions:

$$\begin{aligned}
 \xi_1 &= x \ln x, & \eta_1 &= -2(1 + \ln x); \\
 \xi_2 &= x, & \eta_2 &= -2.
 \end{aligned}$$

This means that Equation 1.34 admits two linearly independent operators

$$X_1 = x \ln x \frac{\partial}{\partial x} - 2(1 + \ln x) \frac{\partial}{\partial y} \quad \text{and} \quad X_2 = x \frac{\partial}{\partial x} - 2 \frac{\partial}{\partial y}. \quad (1.36)$$

Hence the set of all operators admitted by Equation 1.34 is a two-dimensional vector space with the basis given by operators 1.36.

For partial differential equations (PDEs), the construction of an admitted group parallels the ODE development previously described. In the PDE case, one utilizes the prolongation Formulas 1.17 and 1.18 in place of Formulas 1.17' and 1.18'. We illustrate this by the following example.

**Second Example.** Consider the following second-order partial differential equation:

$$u_{xx} + u_{yy} = e^u. \quad (1.37)$$

We seek a symmetry operator

$$X = \xi^1 \frac{\partial}{\partial x} + \xi^2 \frac{\partial}{\partial y} + \eta \frac{\partial}{\partial u},$$

where  $\xi^1$ ,  $\xi^2$ , and  $\eta$  are unknown functions of three variables  $x$ ,  $y$ , and  $u$ . Solution of the determining equation gives that  $\xi^1$  and  $\xi^2$  depend only on  $x$  and  $y$  and satisfy the Cauchy–Riemann system

$$\xi_x^1 - \xi_y^2 = 0, \quad \xi_y^1 + \xi_x^2 = 0, \quad (1.38)$$

while

$$\eta = -2\xi_x^1.$$

So, Equation 1.37 admits the infinite-dimensional vector space of operators

$$X = \xi^1(x, y) \frac{\partial}{\partial x} + \xi^2(x, y) \frac{\partial}{\partial y} - 2\xi_x^1 \frac{\partial}{\partial u}, \quad (1.39)$$

where  $\xi^1$  and  $\xi^2$  are defined by Equations 1.38.

## 1.6. LIE ALGEBRAS

Now we return to the general properties of the determining equation. It can be seen from Equation 1.33 that the determining equation is a linear partial differential equation with unknown functions  $\xi$  and  $\eta$  of two variables  $x$  and  $y$ . It follows that the set of all its solutions is a vector space. However,

there is another property that is intrinsic to determining equations. A set of solutions of any determining equations forms what is called a *Lie algebra*. (This term was introduced by H. Weyl; S. Lie himself used the term *infinitesimal group*.)

We define a *commutator* (Lie bracket)  $[X_1, X_2]$  of operators

$$X_1 = \xi_1 \frac{\partial}{\partial x} + \eta_1 \frac{\partial}{\partial y} \quad \text{and} \quad X_2 = \xi_2 \frac{\partial}{\partial x} + \eta_2 \frac{\partial}{\partial y}$$

by the formula

$$[X_1, X_2] = X_1 X_2 - X_2 X_1, \quad (1.40)$$

or

$$[X_1, X_2] = (X_1(\xi_2) - X_2(\xi_1)) \frac{\partial}{\partial x} + (X_1(\eta_2) - X_2(\eta_1)) \frac{\partial}{\partial y}. \quad (1.40')$$

It follows that the commutator

1. Is bilinear,

$$\begin{aligned} [X, c_1 X_1 + c_2 X_2] &= c_1 [X, X_1] + c_2 [X, X_2] \\ [c_1 X_1 + c_2 X_2, X] &= c_1 [X_1, X] + c_2 [X_2, X] \end{aligned}$$

2. Is skew-symmetric,  $[X_1, X_2] = -[X_2, X_1]$

3. Satisfies the Jacobi identity,

$$[X_1, [X_2, X_3]] + [X_2, [X_3, X_1]] + [X_3, [X_1, X_2]] = 0$$

**Definition 1.5.** A Lie algebra of operators 1.7 is a vector space  $L$  that includes the commutator  $[X_1, X_2]$  along with any two operators  $X_1, X_2 \in L$ . This Lie algebra is denoted by the same letter  $L$ , and the dimension of the Lie algebra is the proper dimension of the vector space  $L$ .

**Remark 1.** Let  $L_r$  be an  $r$ -dimensional vector space with a basis  $X_1, \dots, X_r$ , i.e., any  $X \in L_r$  can be decomposed as follows:

$$X = \sum_{\mu=1}^r C^\mu X_\mu, \quad C^\mu = \text{const.} \quad (1.41)$$

Then it follows from Definition 1.5 that  $L_r$  is a Lie algebra iff

$$[X_\mu, X_\nu] = \sum_{\lambda=1}^r c_{\mu\nu}^\lambda X_\lambda, \quad \mu, \nu = 1, \dots, r, \quad (1.42)$$

where  $c_{\mu\nu}^\lambda$  are real constants called structure constants of  $L_r$ .

**Remark 2.** A Lie algebra  $L_r$  generates an  $r$ -parameter group of transformations 1.1 with a vector-parameter  $a = (a^1, \dots, a^r)$ . To construct the transformations corresponding to this group, it is sufficient to solve the Lie equations for each basis operator of  $L_r$  and take compositions of these  $r$  one-parameter groups.

One of the general results for second-order equations is the following one due to Lie [1891].

**Theorem 1.7.** For any Equation 1.32, the set of all solutions of the determining equation forms a Lie algebra  $L_r$  of dimension  $r \leq 8$ . The maximum dimension  $r = 8$  is realized if and only if Equation 1.32 is linear or can be linearized by a change of variables.

**Example.** The commutator of operators 1.36 equals

$$[X_1, X_2] = -X_2. \quad (1.36')$$

So property 1.42 is valid and therefore the vector space spanned by the operators  $X_1$  and  $X_2$  is a two-dimensional Lie algebra. According to the First Example in Section 1.5,  $X_1$  and  $X_2$  are two linearly independent solutions of the determining equation for Equation 1.34, and any  $X$  admitted by Equation 1.34 is a linear combination of them. It follows from Theorem 1.7 that equation 1.34 cannot be linearized.

Although we discuss above only groups admitted by second-order differential equations, all concepts and algorithms can be generalized to higher-order equations. Moreover, Lie gave a classification of all ordinary differential equations of arbitrary order according to their admitted groups. This classification is based on enumeration of all possible groups of transformations of the plane. A treatment of this classification is given in Lie [1883] and [1884].

Finally, we are in a position to see why the method of the determining equations is not effective for first-order ODEs  $y' = f(x, y)$ . In this case we have the determining equation

$$X_1(y' - f)|_{y'=f} \equiv \eta_x + (\eta_y - \xi_x)f - \xi_y f^2 - \xi f_x - \eta f_y = 0. \quad (1.43)$$

This does not contain the variable  $y'$ ; that is why the split into an overdetermined system does not occur here.

## 1.7. CONTACT TRANSFORMATIONS

In the theory of differential equations as well as in mechanics and geometry along with point transformations, groups of contact (or tangent)

transformations are widely used (Lie [1896a], [1896b]). Here we will discuss at once the multidimensional case with an arbitrary number  $n$  of independent variables  $x = (x^1, \dots, x^n)$  and the one dependent variable  $u$ . (It is worthwhile to mention that for the greater number of the dependent variables there are no contact transformations different from a point ones; the proof can be found, e.g., in Ovsianikov [1978], Section 28.3, in Anderson and Ibragimov [1979], Section 9, or in Ibragimov [1983], Section 14.10.)

Let  $u'$  denote the set of first derivatives  $u_i = \partial u / \partial x^i$ , and consider a one-parameter group of transformations

$$\bar{x}^i = \varphi^i(x, u, u', a), \quad \bar{u} = \psi(x, u, u', a), \quad \bar{u}_i = \omega_i(x, u, u', a) \quad (1.44)$$

in the  $(2n + 1)$ -dimensional space of variables  $(x, u, u')$ . Transformations 1.44 are referred to as contact transformations when  $\bar{u}_i = \partial \bar{u} / \partial \bar{x}^i$ . In terms of infinitesimal transformations,

$$\begin{aligned} \bar{x}^i &\approx x^i + \xi^i(x, u, u')a, & \bar{u} &\approx u + \eta(x, u, u')a, \\ \bar{u}_i &\approx u_i + \zeta_i(x, u, u')a \end{aligned} \quad (1.44')$$

this condition can be rewritten as the prolongation formula

$$\zeta_i = D_i(\eta) - u_j D_i(\xi^j). \quad (1.45)$$

**Theorem 1.8.** The operator

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta \frac{\partial}{\partial u} + \zeta_i \frac{\partial}{\partial u_i} \quad (1.46)$$

is a symbol of a contact transformation group iff

$$\xi^i = -\frac{\partial W}{\partial u_i}, \quad \eta = W - u_i \frac{\partial W}{\partial u_i}, \quad \zeta_i = \frac{\partial W}{\partial x^i} + u_i \frac{\partial W}{\partial u} \quad (1.47)$$

for some function  $W = W(x, u, u')$ . The function  $W$  occurring in this theorem was called by Lie [1896a] the *characteristic function* of the contact transformation group. For ordinary differential equations formulas 1.46 and 1.47 become

$$X = -W_p \frac{\partial}{\partial x} + (W - pW_p) \frac{\partial}{\partial y} + (W_x + pW_y) \frac{\partial}{\partial p}, \quad (1.48)$$

where  $W = W(x, y, p)$  with  $p = y'$ .

**Example.** Consider contact groups admitted by the third-order equation

$$y''' = 0. \quad (1.49)$$

After substitution of operator 1.48 into the determining equation, we find

$$\begin{aligned} W = & C_1 + C_2 x + C_3 x^2 + C_4 y + C_5 p + C_6 xp + C_7(x^2 p - 2xy) \\ & + C_8 p^2 + C_9(xp^2 - 2yp) + C_{10}(x^2 p^2 - 4xyp + 4y^2), \end{aligned} \quad (1.50)$$

where  $C_1, \dots, C_{10}$  are arbitrary constants. Therefore, the substitution of function 1.50 into Equation 1.48 yields 10 linearly independent generators of a Lie algebra  $L_{10}$ :

$$X_1 = \frac{\partial}{\partial y}, \quad X_2 = x \frac{\partial}{\partial y}, \quad X_3 = x^2 \frac{\partial}{\partial y}, \quad X_4 = y \frac{\partial}{\partial y}, \quad X_5 = \frac{\partial}{\partial x},$$

$$X_6 = x \frac{\partial}{\partial x}, \quad X_7 = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}, \quad X_8 = 2p \frac{\partial}{\partial x} + p^2 \frac{\partial}{\partial y},$$

$$X_9 = (y - xp) \frac{\partial}{\partial x} - \frac{1}{2} xp^2 \frac{\partial}{\partial y} - \frac{1}{2} p^2 \frac{\partial}{\partial p},$$

$$X_{10} = \left( xy - \frac{1}{2} x^2 p \right) \frac{\partial}{\partial x} + \left( y^2 - \frac{1}{4} x^2 p^2 \right) \frac{\partial}{\partial y} + \left( yp - \frac{1}{2} xp^2 \right) \frac{\partial}{\partial p}.$$

# Integration of Second-Order Ordinary Differential Equations

## 2.1. SOLVABLE LIE ALGEBRAS AND SUCCESSIVE REDUCTION OF ORDER

As stated in Section 1.4 (see also Dickson [1924], Ovsiannikov [1962], Olver [1986], Bluman and Kumei [1989], Stephani [1989], and Ibragimov [1989a], [1991]), a one-dimensional symmetry algebra can be used to reduce by one the order of a second-order equation. So it is natural to expect that given a two-dimensional algebra the order can be reduced twice, which means we have integrated the equation.

We begin by discussing the properties of Lie algebras that we need in this chapter. Let  $L_r$  be an  $r$ -dimensional Lie algebra with any  $r < \infty$  and let  $N$  be a linear subspace of  $L_r$ .

**Definition 2.1.** The subspace  $N$  is referred to as a subalgebra if  $[X, Y] \in N$  for all  $X, Y \in N$  (i.e., this subspace is an algebra itself), and as an ideal if  $[X, Y] \in N$  for all  $X \in N$  and all  $Y \in L_r$ .

If  $N$  is an ideal, then one can introduce an equivalence relation: Operators  $X$  and  $Y$  from  $L_r$  are said to be equivalent if  $Y - X \in N$ . The set of all operators that are equivalent to a given operator  $X$  is referred to as a coset represented by the operator  $X$ ; any element  $Y$  of this coset has the form  $Y = X + Z$  for some  $Z \in N$ . The set of all cosets is naturally endowed with a Lie algebra structure and is called the quotient algebra of the Lie algebra  $L_r$  by its ideal  $N$ . It is denoted by  $L_r/N$ . One may consider the representatives of appropriate cosets as elements of this quotient algebra.

If all constructions are considered in the complex domain, then the following theorem is valid.

**Theorem 2.1.** In any algebra  $L_r$ ,  $r > 2$ , there exists a two-dimensional subalgebra. Moreover, any operator  $X \in L_r$  can be included in a two-dimensional subalgebra.

**Definition 2.2.** A Lie algebra  $L_r$  is solvable if there is a sequence

$$L_r \supset L_{r-1} \supset \cdots \supset L_1 \quad (2.1)$$

of subalgebras of dimensions  $r, r-1, \dots, 1$ , respectively such that  $L_{s-1}$  is an ideal in  $L_s$  ( $s = 2, \dots, r$ ).

A convenient criterion of solvability is formulated in terms of derived algebras.

**Definition 2.3.** Let  $X_1, \dots, X_r$  be a basis of an algebra  $L_r$ . The linear span of the commutators  $[X_\mu, X_\nu]$  of all possible pairs of the basis operators is an ideal denoted by  $L'_r$  and called the derived algebra. The higher-order derived algebras are defined recursively:  $L_r^{(n+1)} = (L_r^{(n)})'$ ,  $n = 1, 2, \dots$ .

**Theorem 2.2.** An algebra  $L_r$  is solvable if and only if the derived algebra of some order equals zero:  $L_r^{(n)} = 0$  for some  $n > 0$ .

**Corollary.** Any two-dimensional algebra is solvable.

To construct Sequence 2.1 in the case of a two-dimensional algebra  $L_2$ , one may choose a basis  $X_1, X_2$  so that the equality  $[X_1, X_2] = \alpha X_1$  is valid. Then the one-dimensional algebra  $L_1$  spanned by  $X_1$  is an ideal in  $L_2$ , and the quotient algebra  $L_2/L_1$  can be identified with the algebra spanned by  $X_2$ . If this algebra  $L_2$  is admitted by a second-order ODE, one can use the ideal  $L_1$  to reduce the order of this equation by 1 (Sections 1.4.2 and 1.4.3). Further, the resulting first-order equation admits the quotient algebra  $L_2/L_1$  and hence it can be integrated by any of the methods discussed in Section 1.4.

It is clear that an ordinary differential equation of order  $n > 2$  can be integrated by this method of successive reduction of order if it admits a solvable  $n$ -dimensional Lie algebra. For detailed discussions, see Olver [1986], Theorems 2.60, 2.61, and 2.64.

## 2.2. METHOD OF CANONICAL VARIABLES

If a given second-order equation admits a two-dimensional Lie algebra, then instead of using the preceding method of successive reduction of order, one may change variables such that the equation is integrable.

### 2.2.1. CANONICAL FORMS OF $L_2$

To state the basic theorems of this section we need, together with Commutator 1.40, the pseudoscalar (skew) product

$$X_1 \vee X_2 = \xi_1 \eta_2 - \eta_1 \xi_2, \quad (2.2)$$

$$X_1 = \xi_1 \frac{\partial}{\partial x} + \eta_1 \frac{\partial}{\partial y}, \quad X_2 = \xi_2 \frac{\partial}{\partial x} + \eta_2 \frac{\partial}{\partial y}. \quad (2.3)$$

A classification of two-dimensional Lie algebras according to their structural properties is based on the fact that the equations  $[X_1, X_2] = 0$ , and  $X_1 \vee X_2 = 0$  are invariant under both changes of bases in  $L_2$  and changes of the variables  $x$  and  $y$ . Based on this, the set of all two-dimensional Lie algebras is divided into the following inequivalent types:

$$\begin{aligned} \text{I. } [X_1, X_2] &= 0, & X_1 \vee X_2 &\neq 0; \\ \text{II. } [X_1, X_2] &= 0, & X_1 \vee X_2 &= 0; \\ \text{III. } [X_1, X_2] &\neq 0, & X_1 \vee X_2 &\neq 0; \\ \text{IV. } [X_1, X_2] &\neq 0, & X_1 \vee X_2 &= 0. \end{aligned} \quad (2.4)$$

The following theorem states that in each type is only one inequivalent algebraic structure.

**Theorem 2.3.** By a suitable choice of the basis  $X_1, X_2$ , any two-dimensional Lie algebra can be reduced to one of four different types, which are determined by the following canonical structural relations:

$$\begin{aligned} \text{I. } [X_1, X_2] &= 0, & X_1 \vee X_2 &\neq 0; \\ \text{II. } [X_1, X_2] &= 0, & X_1 \vee X_2 &= 0; \\ \text{III. } [X_1, X_2] &= X_1, & X_1 \vee X_2 &\neq 0; \\ \text{IV. } [X_1, X_2] &= X_1, & X_1 \vee X_2 &= 0. \end{aligned}$$

These structural relations are invariant under any change of variables. Using this invariance one can simplify the form of the basis operators  $X_1$  and

$X_2$ . This leads to the following:

**Theorem 2.4.** The basis of an algebra  $L_2$  can be reduced by a suitable change variable of to one of the following forms:

$$\begin{aligned}
 \text{I. } X_1 &= \frac{\partial}{\partial x}, & X_2 &= \frac{\partial}{\partial y}; \\
 \text{II. } X_1 &= \frac{\partial}{\partial y}, & X_2 &= x \frac{\partial}{\partial y}; \\
 \text{III. } X_1 &= \frac{\partial}{\partial y}, & X_2 &= x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}; \\
 \text{IV. } X_1 &= \frac{\partial}{\partial y}, & X_2 &= y \frac{\partial}{\partial y}.
 \end{aligned} \tag{2.5}$$

The variables  $x$  and  $y$  are called canonical variables.

### 2.2.2. INTEGRATION ALGORITHM

Now we identify, by the types described in Theorem 2.4, all second-order ODEs admitting two-dimensional Lie algebras, and then we integrate them.

**Type I.** To construct all second-order equations admitting an algebra  $L_2$  with the basis  $X_1 = \partial/\partial x$ ,  $X_2 = \partial/\partial y$ , one finds a basis for the second-order differential invariants. Prolongations of these operators coincide with the operators themselves; that is why  $y'$  and  $y''$  provide a basis for differential invariants. Hence the general second-order equation admitting  $L_2$  of the first type has the form

$$y'' = f(y'). \tag{2.6}$$

Integration yields

$$\int \frac{dy'}{f(y')} = x + C_1, \quad \text{or equivalently, } y' = \varphi(x + C_1),$$

whence

$$y = \int \varphi(x + C) d(x + C) + C_2.$$

**Type II.** In this case, a basis for the differential invariants is  $x$  and  $y''$ , and the invariant differential equation has the form

$$y'' = f(x). \tag{2.7}$$

Integration yields

$$y = \int \left( \int f(x) dx \right) dx + C_1 x + C_2.$$

**Type III.** In this case, a basis for the differential invariants is  $y'$  and  $xy''$ , and the invariant differential equation has the form

$$y'' = \frac{1}{x} f(y'). \quad (2.8)$$

Integration yields

$$\int \frac{dy'}{f(y')} = \ln x + C_1, \quad \text{or} \quad y' = \varphi(\ln x + C_1),$$

then

$$y = \int \varphi(\ln x + C_1) dx + C_2.$$

**Type IV.** In this case, a basis for the differential invariants is  $x$  and  $y''/y'$ , and the invariant differential equation has the form

$$y'' = f(x) y'. \quad (2.9)$$

Integration yields

$$y = C_1 \int e^{\int f(x) dx} dx + C_2.$$

The preceding results can be implemented in the following five-step algorithm for the integration of second-order ODEs (for details, see Ibragimov [1991], [1992]).

**First Step.** Calculate an admitted Lie algebra  $L_r$ .

**Second Step.** If  $r > 2$ , determine subalgebra  $L_2 \subset L_r$ . If  $r < 2$ , the ODE cannot be completely integrated by the Lie group method.

**Third Step.** Calculate the commutator 1.40' and the pseudoscalar product 2.2 for a basis of  $L_2$ ; if necessary, change the basis in accordance with Theorem 2.3.

**Fourth Step.** Introduce canonical variables in accordance with Theorem 2.4. Rewrite your differential equation in canonical variables and integrate it.

**Fifth Step.** Rewrite the solution in the original variables.

**2.2.3. EXAMPLE**

We apply the algorithm to the equation

$$y'' = \frac{y'}{y^2} - \frac{1}{xy}. \quad (2.10)$$

**First Step.** We solve the determining equation and find that Equation 2.10 admits  $L_2$  spanned by

$$X_1 = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}, \quad X_2 = x \frac{\partial}{\partial x} + \frac{y}{2} \frac{\partial}{\partial y}. \quad (2.11)$$

**Second Step.** Here,  $r = 2$  and therefore we proceed to the third step.

**Third Step.** Here,

$$[X_1, X_2] = -X_1, \quad X_1 \vee X_2 = -\frac{1}{2}x^2y \neq 0.$$

Hence, we conclude from Equations 2.4 that our algebra  $L_2$  is of type III. After changing the sign of  $X_2$ , the algebra spanned by

$$X_1 = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}, \quad X_2 = -x \frac{\partial}{\partial x} - \frac{y}{2} \frac{\partial}{\partial y} \quad (2.11')$$

has the canonical structure given by Theorem 2.3.

**Fourth Step.** Here  $X_1$  is reduced to the operator of translations by the change of variables

$$t = \frac{y}{x}, \quad u = -\frac{1}{x}. \quad (2.12)$$

After this change, the operators become

$$\bar{X}_1 = \frac{\partial}{\partial u}, \quad \bar{X}_2 = \frac{t}{2} \frac{\partial}{\partial t} + u \frac{\partial}{\partial u}. \quad (2.11'')$$

They differ from the canonical form of type III given by Equation 2.5 by a factor of  $\frac{1}{2}$  in  $\bar{X}_2$ . This could be brought to the canonical form. However, Operators 2.11'' are already in a simple form. In order to avoid the singularity  $u' = \infty$ , we exclude the following solutions of Equation 2.10:

$$y = Cx. \quad (2.13)$$

After the change of variables, Equation 2.10 becomes

$$\frac{u''}{u'^2} + \frac{1}{t^2} = 0. \quad (2.10')$$

Integration of Equation 2.10' gives the following two solutions:

$$u = -\frac{t^2}{2} + C \quad (2.14)$$

and

$$u = \frac{t}{C_1} + \frac{1}{C_1^2} \ln |C_1 t - 1| + C_2. \quad (2.15)$$

**Fifth Step.** After we substitute Expressions 2.12 into Equations 2.14 and 2.15 and take into account Equation 2.13, then we obtain the following general solution of the nonlinear Equation 2.10:

$$y = Cx, \quad (2.13)$$

$$y = \pm \sqrt{2x + Cx^2}, \quad (2.16)$$

$$C_1 y + C_2 x + x \ln |C_1 \frac{y}{x} - 1| + C_1^2 = 0, \quad (2.17)$$

where the  $C$ s are arbitrary constants.

# 3

## Group Classification of Second-Order Ordinary Differential Equations

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In Chapter 2 we discussed group theoretical methods for integrating second-order ODEs. The conclusion is that if a second-order equation admits a one-dimensional Lie algebra  $L_1$ , then we can reduce its order by 1 and if it admits a two-dimensional algebra  $L_2$ , we can completely integrate the equation by group methods. This conclusion led to the classification of second-order equations admitting an  $L_2$ . This classification is accomplished by Theorem 2.4. As a result of this classification we obtain four inequivalent types of equations, Formulas 2.6–2.9. Each of these four types involves one arbitrary function of one variable. Further, if one takes into account the allowed arbitrary changes of variables, each type involves two additional arbitrary functions of two variables each. Hence the totality of second-order ODEs integrable by Lie group methods is infinite in the sense just described.

Some of equations appearing in this classification admit higher-dimensional Lie algebras  $L_r$  or  $r$ -parameter local Lie groups,  $2 < r \leq 8$  (see Theorem 1.7). So we come to the problem of the complete group classification of ODEs. Lie [1883] solved this problem for equations of arbitrary order. Here, we only present this classification for second-order equations. This permits us to present the essence of his method and at the same time to be exhaustive.

### 3.1. LIE'S CLASSIFICATION OF EQUATIONS ADMITTING THREE-DIMENSIONAL ALGEBRAS

The group classification of ordinary differential equations is based upon the enumeration of all possible Lie algebras of operators acting on the plane

$(x, y)$ . The basis operators of each algebra are simplified by a suitable change of variables. Algebras connected by a change of variables are called similar. Equations that admit similar algebras are also similar (equivalent) in the sense that they can be transformed into one another by a change of variables.

The classification happens to be of an especially simple form in the case of second-order equations (see Lie [1889], Section 3). In this case the dimension of a maximal admitted algebra has only the values 1, 2, 3, and 8. The dimensions 1 and 2 were discussed in the previous chapters. Here we summarize results for second-order equations admitting a three-dimensional algebra.

Lie gave his classification in the complex domain. The results on the enumeration of all nonsimilar (under complex changes of variables) three-dimensional algebras and of invariant equations are given, e.g., in Lie [1891].

For three-dimensional algebras  $L_3$ , the construction of invariant equations is accomplished by solving the determining equation 1.33 with respect to the unknown function  $f(x, y, y')$ . We illustrate this construction for  $L_3$  spanned by

$$X_1 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \quad X_2 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, \quad X_3 = x^2 \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y}. \quad (3.1)$$

For the operator  $X_1$  we have  $\xi = 1$ ,  $\eta = 1$ . After substitution of these values, Equation 1.33 becomes  $\partial f / \partial x + \partial f / \partial y = 0$ , so that

$$f = f(x - y, y').$$

Substitution of this expression for  $f$  and the coordinates  $\xi = x$  and  $\eta = y$  for  $X_2$  into the determining equation yields  $zf_z + f = 0$ , where  $z = x - y$ . It follows that

$$f = \frac{g(y')}{x - y}. \quad (3.2)$$

Finally, substitution of Function 3.2 and the coordinates  $\xi = x^2$  and  $\eta = y^2$  of  $X_3$  into Equation 1.33 yields

$$2y' \frac{dg}{dy'} - 3g + 2(y'^2 - y') = 0.$$

It follows that

$$g = -2(y' + C^{3/2} + y'^2), \quad C = \text{const.} \quad (3.3)$$

Thus the algebra  $L_3$  is admitted by the equation

$$y'' + 2 \frac{y' + Cy'^{3/2} + y'^2}{x - y} = 0, \quad C = \text{const.}$$

### 3.2. SUMMARY OF LIE'S GENERAL CLASSIFICATION

Lie demonstrated that if a second-order equation admits an algebra  $L_r$  with  $r \geq 4$ , then it necessarily admits an eight-dimensional algebra. However, all such equations are linearized (Theorem 1.7) and they are equivalent to the equation  $y'' = 0$ . This completes Lie's classification of second-order ODEs,  $y'' = f(x, y, y')$ . Namely, if an equation admits  $L_1$ , the action of  $L_1$  can be transformed into translations in  $x$ . Therefore, the canonical form of an equation admitting an  $L_1$  can be taken to be

$$y'' = f(y, y'). \quad (3.4)$$

In the case of  $L_2$ , the corresponding four canonical forms of the invariant equations are given by Equations 2.6–2.9. In the case of  $L_3$ , they are given by the four canonical forms listed in Section 8.4, Equations iv–vii. In the case of higher dimensions, as discussed previously, the only canonical form is

$$y'' = 0. \quad (3.5)$$

Recall that Lie made his classification in the complex domain. If one deals with the real domain only, some of the equations given in Section 8.4 will split into two inequivalent equations (in the sense of real changes of variables). For example, the equations

$$y'' = C(1 + y'^2)^{3/2} e^{q \arctan y'}$$

and

$$\bar{y}'' = C\bar{y}^{(k-2)/(k-1)}$$

are inequivalent in the real domain. However, the first equation is converted into the second with  $k = (q + i)/(q - i)$  via the complex change of variables  $\bar{x} = \frac{1}{2}(y - ix)$ ,  $\bar{y} = \frac{1}{2}(y + ix)$ .

### 3.3. LINEARIZATION

One can extract, from several results of Lie ([1883], [1891]), the following statement (Ibragimov [1991], [1992]).

**Theorem 3.1.** The following assertions are equivalent:

i. A second-order ODE

$$y'' = f(x, y, y') \quad (3.6)$$

can be linearized by a change of variables.

ii. Equation 3.6 has the form

$$y'' + F_3(x, y)y'^3 + F_2(x, y)y'^2 + F_1(x, y)y' + F(x, y) = 0, \quad (3.7)$$

with coefficients  $F_3$ ,  $F_2$ ,  $F_1$ , and  $F$  satisfying the integrability conditions of an auxiliary overdetermined system,

$$\begin{aligned} \frac{\partial z}{\partial x} &= z^2 - Fw - F_1z + \frac{\partial F}{\partial y} + FF_2, \\ \frac{\partial z}{\partial y} &= -zw + FF_3 - \frac{1}{3} \frac{\partial F_2}{\partial x} + \frac{2}{3} \frac{\partial F_1}{\partial y}, \\ \frac{\partial w}{\partial x} &= zw - FF_3 - \frac{1}{3} \frac{\partial F_1}{\partial y} + \frac{2}{3} \frac{\partial F_2}{\partial x}, \\ \frac{\partial w}{\partial y} &= -w^2 + F_2w + F_3z + \frac{\partial F_3}{\partial x} - F_1F_3. \end{aligned} \quad (3.8)$$

iii. Equation 3.6 admits an  $L_8$ .

iv. Equation 3.6 admits an  $L_2$  with a basis  $X_1, X_2$  such that their pseudo-scalar product (Equation 2.2)  $X_1 \vee X_2$  vanishes.

**First Example.** Consider equations of the form (2.6)

$$y'' = f(y'). \quad (3.9)$$

In accordance with Theorem 3.1(ii), it is necessary that the function  $f(y')$  is a polynomial of the third degree, i.e., Equation 3.9 has the form

$$y'' + A_3y'^3 + A_2y'^2 + A_1y' + A_0 = 0 \quad (3.10)$$

with constant coefficients. One can easily verify that the auxiliary system 3.8 for Equation 3.10 is integrable. Therefore Equation 3.10 is linearized.

**Second Example** (see also Mahomed and Leach [1989]). Consider equations of the form,

$$y'' = \frac{1}{x} f(y'). \quad (3.11)$$

Again, by Theorem 3.1(ii), we have to consider only equations of the form

$$y'' + \frac{1}{x} (A_3 y'^3 + A_2 y'^2 + A_1 y' + A_0) = 0,$$

with constant coefficients  $A_i$ . In this case, the integrability conditions of Equations 3.8 yield

$$A_2(2 - A_1) + 9A_0A_3 = 0 \quad \text{and} \quad 3A_3(1 + A_1) - A_2^2 = 0.$$

We put  $A_3 = -a$  and  $A_2 = -b$  and obtain

$$A_1 = -\left(1 + \frac{b^2}{3a}\right) \quad \text{and} \quad A_0 = -\left(\frac{b}{3a} + \frac{b^3}{27a^2}\right).$$

Hence, Equation 3.11 is linearized iff it is of the form

$$y'' = \frac{1}{x} \left[ ay'^3 + by'^2 + \left(1 + \frac{b^2}{3a}\right)y' + \frac{b}{3a} + \frac{b^3}{27a^2} \right]. \quad (3.12)$$

A linearizing change of variables can be found via Theorem 3.1(iv).

For example, we find a linearization of Equation 3.12 in the case  $a = 1$ ,  $b = 0$ , i.e., of the equation

$$y'' = \frac{1}{x} (y' + y'^3). \quad (3.13)$$

This equation admits  $L_2$  with the basis

$$X_1 = \frac{1}{x} \frac{\partial}{\partial x}, \quad X_2 = \frac{y}{x} \frac{\partial}{\partial x}, \quad (3.14)$$

which satisfies the condition  $X_1 \vee X_2 = 0$  of Theorem 3.1(iv). These operators are of type II from Equations 2.4. Therefore a linearization is obtained by employing the canonical variables

$$\bar{x} = y \quad \text{and} \quad \bar{y} = \frac{1}{2}x^2$$

so that

$$\bar{X}_1 = \frac{\partial}{\partial \bar{y}}, \quad \bar{X}_2 = \bar{x} \frac{\partial}{\partial \bar{y}}.$$

Then, excluding the special solution  $y = \text{const.}$ , Equation 3.13 becomes the following linear equation:

$$\bar{y}'' + 1 = 0.$$

**Third Example.** Consider the equation

$$y'' = F(x, y), \tag{3.15}$$

where  $F(x, y)$  is nonlinear in  $y$ .

Equation 3.15 is a particular case of Equation 3.7 with  $F_1 = F_2 = F_3 = 0$ . System 3.8 is written

$$\begin{aligned} z_x &= z^2 + Fw - F_y, & w_x &= zw, \\ z_y &= -zw, & w_y &= -w^2. \end{aligned}$$

One of the integrability conditions of this system is

$$z_{xy} = z_{yx}.$$

It gives the necessary condition for linearization:

$$F_{yy} = 0.$$

Hence Equation 3.15 is not linearizable.

# Invariant Solutions

Lie's investigations were centered on problems of general integrability of differential equations by means of group theory. This is why many of his papers deal with ordinary differential equations or with linear partial differential equations of first order. In his paper on higher-order partial differential equations, Lie [1895] considered solutions invariant under groups admitted by these equations. With the help of these invariant solutions, he returned to the problem of general integrability. However, the majority of the partial differential equations that appear in mathematical physics are not completely integrable by means of group theoretic techniques. In our opinion, this is one of the main reasons why Lie's approach was not widely used and taught in applied mathematics and mathematical physics for a long time.

On the other hand, special types of exact solutions of differential equations were known and successfully used in mechanics and physics. These solutions were found by ad hoc methods. As the result of the efforts of many people, these special solutions are now identified as invariant solutions in the sense of Lie. This connection with group theory led to the increased activity in group analysis of partial differential equations that occurred after 1950 (strongly influenced by the books of Birkhoff [1950], Sedov [1957], Petrov [1961], Ovsiannikov [1962], and Ames [1965], [1972]) and eventually to the explosion of activity starting in about 1970.

This chapter provides an introduction to the notion of an invariant solution and the construction of sets of independent invariant solutions (optimal systems in the sense of Ovsiannikov [1962]). It is sufficient, for purposes of illustration, to consider ordinary differential equations.

## 4.1. NOTION OF AN INVARIANT SOLUTION

In order to explain the notion of an invariant solution, we consider the simple example of the Riccati equation from Section 1.4.1,

$$y' + y^2 = 2/x^2. \quad (4.1)$$

One of its symmetry groups is given by

$$\bar{x} = xe^a, \quad \bar{y} = ye^{-a}. \quad (4.2)$$

We choose a particular solution of Equation 4.1, say,

$$y = \frac{2x^3 + 1}{x(x^3 - 1)}. \quad (4.3)$$

Transformation 4.2 converts this solution into a one-parameter family of solutions

$$\bar{y} = \frac{2\bar{x}^3 + C}{\bar{x}(\bar{x}^3 - C)} \quad (4.4)$$

with  $C = e^{3a}$ . Here  $C > 0$  because  $-\infty < a < \infty$ .

How general is the solution given by Equation 4.4? The solutions with  $C \leq 0$  and  $C = \infty$  are missing. Is it possible to recover those with  $C < 0$  by using the reflection  $x \mapsto -x$ ,  $y \mapsto -y$ , which is a discrete symmetry of Equation 4.1.

What about the limiting cases  $C = 0$  and  $C = \infty$ ? It turns out that the solutions

$$y_1 = 2/x \quad \text{and} \quad y_2 = -1/x, \quad (4.5)$$

which correspond to  $C = 0$  and  $C = \infty$ , respectively, have a special property with respect to the dilations. Namely, each one is converted into itself, i.e.,

$$\bar{y}_1 = 2/\bar{x} \quad \text{and} \quad \bar{y}_2 = -1/\bar{x}, \quad (4.5')$$

and hence they are invariant in the sense of Definition 1.3. So, Solutions 4.5 are called invariant solutions. More precisely, Solutions 4.5 are invariant solutions of Equation 4.1 under the action of the dilation group given by Equations 4.2.

In general, if a solution is invariant, in the sense of Definition 1.3, under the action of any symmetry group, it is called a *group invariant solution* or, for brevity, an *invariant solution*.

Now, we illustrate a technique for constructing all invariant solutions corresponding to a symmetry group when its infinitesimal operators are known. We do this for Equation 4.1, using the infinitesimal operator

$$X = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} \quad (4.6)$$

of Transformations 4.2.

By Theorem 1.3, any invariant solution can be written in terms of invariants. In our case the only independent invariant is  $J = xy$ . Therefore all invariant solutions can be written as

$$J = \text{const.}, \quad \text{i.e.,} \quad xy = C.$$

Substitution of the expression  $y = C/x$  into Equation 4.1 yields  $C^2 - C - 2 = 0$ . Thus we have  $C = 2$  and  $C = -1$ . As a result we obtain two solutions,

$$y_1 = 2/x \quad \text{and} \quad y_2 = -1/x.$$

These are precisely the two invariant solutions given by Equations 4.5 and there are no other invariant solutions of Equation 4.1 with respect to the dilations 4.2.

## 4.2. OPTIMAL SYSTEMS

In the preceding section, we considered invariant solutions with respect to a given one-parameter group or one-dimensional Lie algebra. If a differential equation admits a Lie algebra  $L_r$  of dimension  $r > 1$ , one could in principle consider invariant solutions based on one, two, etc., dimensional subalgebras of  $L_r$ . However, there are an infinite number of subalgebras, e.g., one-dimensional subalgebras. This problem becomes manageable by recognizing that if two subalgebras are similar, i.e., they are connected with each other by a transformation from the symmetry group (with Lie algebra  $L_r$ ), then their corresponding invariant solutions are connected with each other by the same transformation. Therefore, it is sufficient to put into one class all similar subalgebras of a given dimension, say  $s$ , and select a representative from each class. The set of these representatives of all these classes is called an *optimal system of order s* (Ovsiannikov [1962], [1978]). In order to find all invariant solutions with respect to  $s$ -dimensional subalgebras, it is sufficient to construct invariant solutions for the optimal system of order  $s$ . The set of invariant solutions obtained in this way is called an *optimal system of invariant solutions*. Of course the form of these invariant solutions depends on the choice of representatives.

It is sufficient to illustrate this construction for  $r = 2$ . Let us return to the example in Section 2.2.3. It was shown there that the equation

$$y'' = \frac{y'}{y^2} - \frac{1}{xy} \tag{4.7}$$

admits a two-parameter group  $G_2$  with the Lie algebra  $L_2$  spanned by

$$X_1 = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}, \quad X_2 = x \frac{\partial}{\partial x} + \frac{y}{2} \frac{\partial}{\partial y}. \quad (4.8)$$

We will also use the fact that Equation 4.7 has the reflection symmetry  $y \mapsto -y$ . Let us find the optimal system of one-dimensional subalgebras and the corresponding invariant solutions.

Because we are interested in one-dimensional subalgebras, we determine how an arbitrary operator

$$X = \sum_{\mu=1}^2 C^\mu X_\mu \quad (4.9)$$

of the algebra  $L_2$  transforms under  $G_2$ . Transformations of  $G_2$  can be obtained as a composition of two one-parameter groups, one generated by  $X_1$  and the other by  $X_2$ . Therefore, we need only consider transformations of Operators 4.9 under the action of these two one-parameter groups separately.

The operator  $X_1$  generates the group of projective transformations

$$\bar{x} = \frac{x}{1 - a_1 x}, \quad \bar{y} = \frac{y}{1 - a_1 x}, \quad (4.10)$$

with parameter  $a_1$ . Using Equation 1.13, we find that under these transformations Operator 4.9 becomes

$$\bar{X} = \sum_{\mu=1}^2 C^\mu \bar{X}_\mu, \quad (4.11)$$

where

$$\bar{X}_1 = X_1, \quad \bar{X}_2 = X_2 - a_1 X_1. \quad (4.12)$$

In the original basis, Operator 4.11 is rewritten as

$$\bar{X} = \sum_{\mu=1}^2 \bar{C}^\mu X_\mu. \quad (4.13)$$

It follows from Equations 4.11–4.13 that

$$\bar{C}^1 = C^1 - a_1 C^2, \quad \bar{C}^2 = C^2. \quad (4.14)$$

Similar calculations with the group of dilations

$$\bar{x} = a_2 x, \quad \bar{y} = \sqrt{a_2} y \quad (a_2 > 0) \quad (4.15)$$

generated by the operator  $X_2$  yield

$$\bar{X}_1 = a_2 X_1, \quad \bar{X}_2 = X_2, \quad (4.16)$$

or, equivalently,

$$\bar{C}^1 = a_2 C^1, \quad \bar{C}^2 = C^2, \quad (4.17)$$

To construct the optimal system of order 1, we must partition Operators 4.9, or equivalently, their coordinate vectors

$$C = (C^1, C^2), \quad (4.18)$$

into similarity classes with respect to Transformations 4.14 and 4.17. Following Ovsiannikov [1962], we proceed to solve this problem in a way that is particularly effective for low-dimensional algebras.

In our case we first observe that  $C^2$  is invariant with respect to Transformations 4.14 and 4.17. Further, the transformation of the coordinate  $C^1$  in Transformation 4.14 suggests that we distinguish the cases  $C^2 = 0$  and  $C^2 \neq 0$ .

All Vectors 4.18 with  $C^2 = 0$  are spanned by the unit vector

$$(1, 0). \quad (4.19)$$

Any vector with  $C^2 \neq 0$  can be transformed to the form  $(0, C^2)$  by applying Transformation 4.14 with  $a_1 = C^1/C^2$ . These latter vectors are spanned by the unit vector

$$(0, 1). \quad (4.20)$$

Hence any Operator 4.9 is similar to  $X_1$  or  $X_2$ . Thus the optimal system of order 1 is

$$\{X_1, X_2\}. \quad (4.21)$$

In order to find the corresponding optimal system of invariant solutions we have only to construct the invariant solutions for  $X_1$  and  $X_2$ . These solutions are further simplified by using Dilations 4.15 and the reflection symmetry  $y \rightarrow -y$  of Equation 4.7. As a result we obtain the following optimal system of invariant solutions:

$$y_1 = x, \quad y_2 = \sqrt{2}x. \quad (4.22)$$

The application of Transformations 4.10, 4.15, and reflection to Solutions 4.22 yields the totality of invariant solutions of Equation 4.7:

$$y = Cx \quad (4.23)$$

and

$$y = \pm \sqrt{2x + Cx^2}. \quad (4.24)$$

According to the terminology of Lie group theory, the action of the group  $G_2$  on  $L_2$  given by Equations 4.12 and 4.16 (or, equivalently, by Equations 4.14 and 4.17) is called the *adjoint representation* of  $G_2$ .

The fact that the number of elements in the optimal system of Subalgebras 4.21 and the optimal system of Solutions 4.22 is equal to the dimensionality of the symmetry algebra is an accident of this example and is not true in general. For example, consider the equation

$$y'' = y^{-3}. \quad (4.25)$$

It admits the three-parameter group  $G_3$  with the Lie algebra  $L_3$  spanned by

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = x \frac{\partial}{\partial x} + \frac{1}{2} y \frac{\partial}{\partial y}, \quad X_3 = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}.$$

In this case the optimal system of order 1 is

$$\{X_1, X_2, X_3, X_1 + X_3, X_1 - X_3\}, \quad (4.26)$$

whereas the corresponding optimal system of invariant solutions consists only of one element, namely,

$$y = \sqrt{1 + x^2}. \quad (4.27)$$

Furthermore, application of the group  $G_3$  and the reflection symmetry  $y \mapsto -y$  to Solution 4.27 yields the following two-parameter family of invariant solutions:

$$y = \frac{1}{b} \sqrt{1 + b^4(x + a)^2}, \quad a, b \text{ arbitrary constants.} \quad (4.28)$$

Moreover, this is the general solution of Equation 4.25; hence all solutions are invariant solutions (see Ibragimov [1991] and [1992] for details).

# II

## Generalizations

The question of generalizing Lie's theory of point and contact transformation groups led naturally to developments in two directions: group actions and their generators (vector fields).

This problem was raised by Lie himself in his fundamental work (Lie [1874], page 223) on contact transformations and was formulated as follows:

1. Are there transformations, other than contact transformations, for which tangency of higher order is an invariant condition?
2. Do partial differential equations of order higher than one admit transformations that are not contact transformations?

One positive answer to Lie's questions lies in the introduction of formal transformation groups that act necessarily on infinite-dimensional spaces. These formal groups are called Lie–Bäcklund transformation groups.

Another positive answer to the second question (not the first) is based on the introduction of transformations that act only on the family of solutions of a given differential equation and are called in the literature Bäcklund transformations. These transformations do not have the group property and are not connected in general with Lie–Bäcklund transformation groups. Nevertheless, for evolution equations and those that can be rewritten in this form, Bäcklund transformations can be represented as invariants of Lie–Bäcklund groups defined by the evolution equations (considered as Lie equations) (Fokas and Anderson [1979]; Anderson and Ibragimov [1978]).

For the purposes of applications, what has proven to be most expedient are Bäcklund transformations and infinitesimal generators of Lie–Bäcklund groups.

In Part II, we give a concise presentation of a theory of Lie–Bäcklund transformation groups as well as closely related generalizations.

# 5

## Lie–Bäcklund Transformation Groups

### 5.1. STEPS TOWARD A THEORY

In this section we attempt to sketch the crucial ideas and key applications that led to a theory of Lie–Bäcklund transformation groups. The subsections reflect a chronological ordering. We also point out the importance of the order of influence, which starts with key applications, on the development of a theory. However, the main emphasis is on results.

#### 5.1.1. LIE AND BÄCKLUND

In order to explain the work of Lie and Bäcklund, it is important to distinguish between those requirements placed on transformations that arise from tangency conditions alone and those that follow from the group properties. Hereafter, we employ the notation  $u, u_1, \dots$ , for the sets of first-order, second-order, etc. partial derivatives  $\{u_i\}, \{u_{ij}\}, \dots$ .

Lie contact transformations discussed in Section 1.7 can be considered as transformations

$$\bar{x}^i = f^i(x, u, u_1), \quad \bar{u} = f(x, u, u_1), \quad \bar{u}_i = \varphi_i(x, u, u_1) \quad (5.1)$$

that leave invariant the first-order tangency condition

$$du - u_i dx^i = 0. \quad (5.2)$$

In the case of one-parameter groups of contact transformations 1.44, this is

Note that in Lie-Bäcklund theory this is what is called a canonical Lie-Bäcklund operator, and any generator

$$X = \xi^i(x, u, u_1, \dots, u_k) \frac{\partial}{\partial x^i} + \eta^\alpha(x, u, u_1, \dots, u_k) \frac{\partial}{\partial u^\alpha} \quad (5.6)$$

of a formal one-parameter Lie-Bäcklund group can be reduced to the canonical form 5.5 (see Section 5.3).

The results of Johnson's first paper that are relevant to our considerations are as follows: (i) prolongations of Operators 5.5,

$$X = \eta^\alpha \frac{\partial}{\partial u^\alpha} + D_i(\eta^\alpha) \frac{\partial}{\partial u_i^\alpha} + \dots + D_{i_1} \dots D_{i_j}(\eta^\alpha) \frac{\partial}{\partial u_{i_1 \dots i_j}^\alpha}, \quad (5.7)$$

where  $D_i$  is the total derivative with respect to  $x^i$ , i.e.,

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \dots; \quad (5.8)$$

(ii) the definition of the Lie bracket  $[X_1, X_2]$  of these operators; and (iii) the proof of the theorem that the linear space of all operators 5.7 is a Lie algebra (see Johnson [1964a], Definition 2 and Lemma 1, and Definition 3, Theorem 1).

He also notes in this paper that the Lie equations

$$\frac{\partial \bar{u}^\alpha}{\partial a} = \eta^\alpha(x, \bar{u}, \bar{u}_1, \dots, \bar{u}_k), \quad \alpha = 1, \dots, m, \quad (5.9)$$

are not generally solvable for  $k > 1$  even in the analytic case. This is in contrast to Lie's point and contact transformation groups, where  $k = 0$  or 1 and Equations 5.9 are locally solvable.

In his second paper, Johnson [1964b] defines invariance of systems of partial differential equations under Operators 5.5 and derives the determining equation for this invariance. Furthermore, he generalizes Lie's result that the totality of solutions of the determining equation is a Lie algebra (Johnson [1964b], Theorem 2).

Johnson treated this generalization in the framework of Ehresmann's [1953] jet bundles in order to formulate his results in a coordinate-independent form. He used the concept of formal tangent vectors introduced by Hermann [1961] and called the generalized operator 5.5 a  $k$ -vector field. Note that Johnson uses the Kuranishi symbol  $\partial_i^\#$  for  $D_i$  given by Equation 5.8. For an exposition in the context of infinite jet bundles, see Vinogradov [1979] and Kupershmidt [1980].

### 5.1.3. KEY APPLICATIONS

Interest in explaining so-called hidden and dynamical symmetries of physical systems and their consequences within the framework of Lie theory of differential equations led Khukhunashvili [1971] and Anderson, Kumei, and Wulfman [1972a], [1972b], independently, to the introduction of higher derivatives in Lie's infinitesimal techniques and to the rediscovery of Johnson's  $k$ -vector fields. For example, in both papers the Runge–Lenz vector in the  $\mathfrak{so}(4)$  symmetry algebra of the hydrogen atom (Fock [1935]) was identified as a non-Lie operator 5.5. The  $\mathfrak{so}(4, 2)$  dynamical symmetry of the hydrogen atom (Barut and Kleinert [1967], Fronsdal [1967]) was identified by Anderson, Kumei, and Wulfman ([1972a], [1972b]), in their study of the time-dependent Schrödinger equation, as a non-Lie symmetry.

Widespread interest in generalized group analysis of differential equations came from a distinctly different direction, namely, from soliton physics (Zabusky and Kruskal [1965]). It was recognized early that non-Lie symmetries account for the infinite number of conservation laws that the first soliton-type equations possessed, as well as providing a characterization of their soliton solutions as invariant solutions under the action of these symmetries. This led to a flurry of activity in generalized group analysis as a tool to search for more soliton-type equations.

For our purposes it is important to understand that Lie's infinitesimal techniques apply when using the generalization with one caveat, namely, one must a priori fix the order  $k < \infty$  (see, e.g., Kumei [1975], [1977]). A new technique that was not in Lie's theory has proved to be particularly effective for constructing hierarchies of symmetries. It is based on the notion of a *recursion operator* discovered by Lenard for the Korteweg–de Vries equation (see Gardner, Greene, Kruskal, and Miura [1974], Olver [1977], and Magri [1978], [1980]). These recursion operators can be precisely associated with the linear part of Lie–Bäcklund symmetries for evolution equations (Ibragimov and Shabat [1980a], [1980b]; see also Ibragimov [1983], Section 19). This latter association provides a test whether a given evolution equation possesses nontrivial Lie–Bäcklund symmetries.

### 5.1.4. IMPOSITION OF THE GROUP STRUCTURE ON INFINITE-ORDER TANGENT TRANSFORMATIONS

Now we return to the central problem of this chapter: How do we wed infinite-order tangency with group properties? These are two different requirements to impose upon any transformation. The problem is not with the existence of infinite-order tangent transformations. Many examples are given by transformations of the type

$$\bar{x} = f(x, u, \dots, u_s), \quad \bar{u} = \varphi(x, u, \dots, u_k) \quad (5.10)$$

together with their prolongation to all derivatives according to Formulas 1.15, 1.16, etc. The difficulty lies in imposing the group property. For example, in the case of Transformations 5.10 we know transformation groups of this type only for  $k, s \leq 1$  (these are Lie point and contact transformations). However, for  $k$  and/or  $s > 1$ , we do not know of any example of groups given by Equations 5.10. This brings us to a fundamental difficulty. Namely, the problem of wedding infinite-order tangency with the group property leads, for Operators 5.6, to *Lie-Bäcklund equations*

$$\begin{aligned}\frac{\partial \bar{x}^i}{\partial a} &= \xi^i(\bar{x}, \bar{u}, \dots, \bar{u}_s), \\ \frac{\partial \bar{u}^\alpha}{\partial a} &= \eta^\alpha(\bar{x}, \bar{u}, \dots, \bar{u}_k), \\ &\vdots\end{aligned}\tag{5.11}$$

which are not of the Cauchy-Kovalevskaya type and hence their integrability is not assured.

However, there are examples, beyond Lie's point and contact transformation groups, for which one can prove the existence and uniqueness for the Lie-Bäcklund Equations 5.11. For instance, examples are provided by taking  $\xi^i$  arbitrary and  $\eta^\alpha = \xi^i u_i^\alpha$  in Operator 5.6, i.e.,

$$X = \xi^i D_i,\tag{5.12}$$

where  $D_i$  is given by Equation 5.8. The resulting Lie-Bäcklund Equations 5.11 are not of the Cauchy-Kovalevskaya type. However, there is a generalization of the Cauchy-Kovalevskaya theorem, namely, Ovsianikov's theorem in scales of Banach spaces (Ovsianikov [1971]), that is applicable to this example (Ibragimov [1977], Anderson and Ibragimov [1977]). Another class of examples is provided by Operators 5.5 of the form

$$X = u \frac{\partial}{p \partial u}.$$

Here, e.g., for  $p = 2$ , the solvability of the corresponding Lie-Bäcklund equations can be provided only for functions  $u(x)$  from the Jevrey class  $C^{1/2}$  (Anderson and Ibragimov [1979]), Section 12; for arbitrary  $p$  see Ibragimov [1983], Section 16.3).

These examples, as well as those from quantum mechanics, make it clear that there is no hope of proving, in general, the local solvability of the Lie-Bäcklund equations as is the case with the Lie equations. Therefore the problem is divided into two parts. First, candidates for Lie-Bäcklund groups must be identified by using formal power series. The algebraic part was done by Ibragimov [1979] (see also Ibragimov [1983]) for all Lie-Bäcklund Operators 5.6 in a universal space  $[[\mathcal{A}]]$  and is described next, in Section 5.2. The

second part, i.e., the convergence of these formal power series, cannot be universally solved and must be treated separately for each type of Lie-Bäcklund operator.

## 5.2. FORMAL LIE-BÄCKLUND TRANSFORMATION GROUPS

Consider functions  $f(x, u, u_1, \dots, u_k)$  of a finite number of arguments ( $k < \infty$ ) that are locally analytic in the sense that  $f$  is locally expandable in a Taylor series with respect to all arguments. These functions are generalizations of Ritt's *differential polynomials* (Ritt [1950]) and are called *differential functions of finite order* (Ibragimov [1981], [1983]), where the order refers to the highest derivative appearing in  $f$ . The space of all differential functions of all finite orders is denoted by  $\mathcal{A}$ . This space is a vector space with respect to the usual addition of functions and becomes an associative algebra if multiplication is defined by the usual multiplication of functions. Furthermore, it has the important property of being closed under the derivation given by Equation 5.8.

Consider operators of the form

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}, \quad \xi^i, \eta^\alpha \in \mathcal{A}. \quad (5.13)$$

Their prolongation to all derivatives is again denoted by  $X$  and is given by

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha} + \zeta_{i_1 i_2}^\alpha \frac{\partial}{\partial u_{i_1 i_2}^\alpha} + \dots, \quad (5.14)$$

where

$$\begin{aligned} \zeta_i^\alpha &= D_i(\eta^\alpha - \xi^j u_j^\alpha) + \xi^j u_{ji}^\alpha, \\ \zeta_{i_1 i_2}^\alpha &= D_{i_1} D_{i_2}(\eta^\alpha - \xi^j u_j^\alpha) + \xi^j u_{ji_1 i_2}^\alpha, \\ &\dots \end{aligned} \quad (5.15)$$

**Definition 5.1.** An operator given by Equations 5.14 and 5.15, where  $\xi^i$  and  $\eta^\alpha$  are differential functions of finite order, i.e.,  $\xi^i, \eta^\alpha \in \mathcal{A}$ , is called a Lie-Bäcklund operator. The abbreviated operator 5.13 is also referred to as a Lie-Bäcklund operator.

Denote by  $[[\mathcal{A}]]$  the space of formal power series in one symbol  $a$  with coefficients in  $\mathcal{A}$ . The following theorem due to Ibragimov [1979], [1983]

states that any Lie-Bäcklund operator generates a *formal one-parameter group* of tangent transformations acting in  $[[\mathcal{A}]]$ .

**Theorem 5.2.** Given any Lie-Bäcklund operator 5.13, there exists a unique solution of the Lie-Bäcklund equations

$$\begin{aligned} \frac{\partial \bar{x}^i}{\partial a} &= \xi^i(\bar{x}, \bar{u}, \dots, \bar{u}_l), & \bar{x}^i|_{a=0} &= x^i, \\ \frac{\partial \bar{u}^\alpha}{\partial a} &= \eta^\alpha(\bar{x}, \bar{u}, \dots, \bar{u}_k), & \bar{u}^\alpha|_{a=0} &= u^\alpha, \\ &\dots \end{aligned} \tag{5.16}$$

in the space  $[[\mathcal{A}]]$ . This solution is given by formal power series, i.e.,

$$\begin{aligned} \bar{x}^i &= x^i + \sum_{s=1}^{\infty} A_s^i a^s, \\ \bar{u}^\alpha &= u^\alpha + \sum_{s=1}^{\infty} B_s^\alpha a^s, \\ &\dots \end{aligned} \tag{5.17}$$

with coefficients  $A_s^i, B_s^\alpha \in \mathcal{A}$ , where  $A_1^i = \xi^i(x, u, \dots, u_l)$  and  $B_1^\alpha = \eta^\alpha(x, u, \dots, u_k)$ . It is a formal one-parameter group that leaves invariant the infinite-order tangency conditions

$$du^\alpha - u_j^\alpha dx^j = 0, \quad du_i^\alpha - u_{ij}^\alpha dx^j = 0, \dots \tag{5.18}$$

**Definition 5.2.** The formal groups obtained in Theorem 5.2 are called formal Lie-Bäcklund transformation groups. Any formal Lie-Bäcklund transformation group is given by a formal power series and is generated by a Lie-Bäcklund operator (Definition 5.1).

**Definition 5.3.** The formal Lie-Bäcklund transformation group in Definition 5.2 is a local one-parameter group if the series in Equations 5.17 converge, and is called a one-parameter Lie-Bäcklund transformation group.

### 5.3. LIE-BÄCKLUND SYMMETRIES OF DIFFERENTIAL EQUATIONS

Lie's determining equations (see Section 1.5) naturally generalize to formal Lie-Bäcklund symmetry groups (see Johnson [1964b], Ibragimov and

Anderson [1977], and Ibragimov [1983]). However, it is important to note that one must use Definition 1.4 of an admissible group. This is because Definition 1.4' is not applicable for Lie-Bäcklund transformation groups because they are given by formal power series and do not convert classical solutions into classical solutions. This is not to be confused with the fact that in particular cases where one can prove convergence of the power series in Equations 5.17, Definition 1.4' is also applicable.

**Theorem 5.3** (Ibragimov [1979], [1983]). Consider a  $k$ th-order differential equation

$$F(x, u, u_1, \dots, u_k) = 0, \quad (5.19)$$

where  $F \in \mathcal{A}$  is a differential function of order  $k$ . Denote by  $[F]$  the frame of Equation 5.19 given by this equation and all its differential consequences (cf. Introduction to Part I). Then Equation 5.19 admits, in the sense of Definition 1.4, a formal Lie-Bäcklund transformation group generated by a Lie-Bäcklund operator  $X$  iff

$$XF|_{[F]} = 0. \quad (5.20)$$

Equation 5.20 is called the determining equation of the Lie-Bäcklund symmetry group.

It is an immediate consequence of the determining equation that every Lie-Bäcklund operator of the form

$$X_* = \xi^i D_i, \quad \xi^i \in \mathcal{A}, \quad (5.21)$$

is admitted by any differential equation. We remark that Lie-Bäcklund equations for operators 5.21 are solvable (in suitable scales of Banach spaces), as was noted in Section 5.1.4. Moreover the set of all Operators 5.21 is an ideal in the Lie algebra of all Lie-Bäcklund operators. Therefore, it is sufficient to consider the quotient algebra, i.e., to consider Lie-Bäcklund operators with  $\xi^i = 0$ ,

$$X = \eta^\alpha \frac{\partial}{\partial u^\alpha}, \quad \eta^\alpha \in \mathcal{A}, \quad (5.22)$$

which are called *canonical Lie-Bäcklund operators*. In particular, every Operator 5.13 is transformed to the canonical form by the prescription

$$\bar{X} = X - \xi^i D_i.$$

All of these considerations obviously generalize to systems of differential equations.

**Example.** The time-independent Schrödinger equation for the bound states of the hydrogen atom

$$\frac{1}{2} \Delta u + \left( \frac{1}{r} - \mathbf{E} \right) u = 0,$$

where  $\mathbf{E}$  is a positive constant, admits three nontrivial Lie-Bäcklund operators (the Runge-Lenz vector mentioned in Section 5.1.3)

$$X_{(l)} = \frac{\partial}{\partial x^l} + \left[ \frac{u}{r} x^l + \sum_{i \neq l} (x^l u_{ii} - x^i u_{li}) \right] \frac{\partial}{\partial u}, \quad l = 1, 2, 3,$$

or, in the canonical form,

$$X_{(l)} = \left[ \frac{u}{r} x^l - u_l + \sum_{i \neq l} (x^l u_{ii} - x^i u_{li}) \right] \frac{\partial}{\partial u}.$$

also satisfies

$$\frac{dT}{dt} = 0$$

under the usual assumption that  $\rho$  vanishes at infinity. The conserved density  $\tau$  is a function of  $t, x$ , dependent variables, and their partial derivatives up to some order, whereas  $\rho = (\rho^1, \dots, \rho^p)$  is a vector with the same arguments as  $\tau$ . Because the differentiation with respect to  $t$  and  $x$  must be understood as a total differentiation, Equation 6.3 means

$$D_t(\tau) + \sum_{i=1}^p D_i(\rho^i) = 0. \quad (6.3')$$

For an arbitrary differential equation, we do not distinguish space and time variables and denote by  $x = (x^1, \dots, x^n)$  all independent variables. In this case a conservation equation is written in the form

$$D_i(T^i) = 0, \quad (6.4)$$

where  $T^i \in \mathcal{A}$  (i.e., each  $T^i$  is a differential function of finite order, see Section 5.2).

**Definition 6.1.** Equation 6.4 is called a conservation law for Equation 6.1 if the equation

$$D_i \left[ T^i \left( x, u(x), u_1(x), \dots, u_l(x) \right) \right] = 0 \quad (6.5)$$

is satisfied for all solutions  $u(x)$  of Equation 6.1. The vector  $T = (T^1, \dots, T^n)$  is called a conserved vector.

Immediately the question arises, how does one find conservation laws for a given Equation 6.1, namely, how does one find  $T$ ? Although Definition 6.1 follows naturally from physical considerations, technically its applicability depends on knowledge of the solutions of Equation 6.1. Here we arrive at the same situation we encountered in the definition of symmetry groups (cf. Introduction to Part I and the passage from Definition 1.4' to Definition 1.4 in Section 1.3). Namely, we make these considerations algorithmic if we replace the notion of "all solutions" in this definition with the notion of the frame of the differential equation. Because of the presence of higher derivatives in the  $T^i$ 's, we must use the frame mentioned in Theorem 5.3, which is given by Equation 6.1 together with its differential consequences.

In order not to get into an extremely technical discussion of what is meant by the frame of Equation 6.1 in the general case, let us consider only

conservation laws of a fixed order  $l$ . By this we mean that  $l$  is the maximum of the orders of all the  $T^i$  in Equation 6.4. As a result, the left-hand side of Equation 6.4 involves derivatives of order  $\leq l + 1$ . Further, let us restrict ourselves to the case of one equation (the generalization to systems of differential equations is obvious). It follows that if  $s$  is defined by

$$s = \max\{k, l + 1\}, \quad (6.6)$$

then all constructions take place in the finite-dimensional space of variables

$$(x, u, u_1, \dots, u_s). \quad (6.7)$$

More precisely, the frame  $[F]$ , in this case, is a surface given by the following equations

$$\begin{aligned} F &= 0, & \text{if } s = k, \\ F &= 0, \quad D_1 F = 0, \dots, D_{i_1} \cdots D_{i_{s-k}} F = 0, & \text{if } s > k. \end{aligned} \quad (6.8)$$

**Definition 6.2.** Equation 6.4 is called a conservation law for Equation 6.1 if

$$D_i(T^i)|_{[F]} = 0. \quad (6.9)$$

Definition 6.2 brings us to the same situation we encountered when calculating symmetries by using determining equations (see Section 1.5 and compare Equation 6.9 with Equation 5.20). Namely, Equation 6.9 is linear in the unknowns  $T^i$  and splits because it contains derivatives of higher order than those in the  $T^i$ . In principle, we can solve Equation 6.9 and find all  $l$ th order conservation laws for Equation 6.1. Hence we call Equation 6.4 the determining equation for  $l$ th order conservation laws obeyed by Equation 6.1.

This algorithmic possibility was employed by Laplace some 200 years ago. He (see Laplace [1798], Livre II, Chapitre III no. 18) used an analog of Equation 6.9 to derive conservation laws quadratic in velocities for the Kepler problem. In particular he found three new conserved quantities,

$$A_k = m(|v|^2 x^k - (x \cdot v)v^k) + \alpha \frac{x^k}{r}, \quad 1, 2, 3, \quad (6.10)$$

for the equation of motion

$$m\ddot{x} = \alpha \frac{x}{r^3}, \quad \alpha = \text{const.} \quad (6.11)$$

Here  $x = (x^1, x^2, x^3)$ ,  $r = |x|$ ,  $v = \dot{x} \equiv dx/dt$ , and  $x \cdot v = \sum_{l=1}^3 x^l v^l$ . The

It is convenient to rewrite any Lie infinitesimal operator (infinitely prolonged) and the Lie-Bäcklund operator 5.14 in the form

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} \left[ D_{i_1} \cdots D_{i_s}(W^\alpha) + \xi^j u_{ji_1 \dots i_s}^\alpha \right] \frac{\partial}{\partial u_{i_1 \dots i_s}^\alpha}, \quad (6.16)$$

where

$$W^\alpha = \eta^\alpha - \xi^j u_j^\alpha. \quad (6.17)$$

The following operators are associated with any operator 6.16:

$$N^i = \xi^i + W^\alpha \frac{\delta}{\delta u_i^\alpha} + \sum_{s \geq 1} D_{i_1} \cdots D_{i_s}(W^\alpha) \frac{\delta}{\delta u_{ii_1 \dots i_s}^\alpha}, \quad (6.18)$$

where  $\delta/\delta u_i^\alpha$ , etc. denote the corresponding variational derivatives (cf. Equation 6.15), e.g.,

$$\frac{\delta}{\delta u_i^\alpha} = \frac{\partial}{\partial u_i^\alpha} + \sum_{s \geq 1} (-1)^s D_{j_1} \cdots D_{j_s} \frac{\partial}{\partial u_{ij_1 \dots j_s}^\alpha}, \quad (6.19)$$

Operators 6.18 were introduced and called Noether operators by Ibragimov (see, e.g., Ibragimov [1983], Section 22.1).

Even though the three operators (6.15), (6.16), and (6.18) are represented as infinite formal sums they act properly on the space  $\mathcal{A}$ . Namely, they truncate while acting on any differential function.

Further and most importantly they are connected by the following theorem.

**Theorem 6.1** (Ibragimov [1983]). Operators 6.15, 6.16, and 6.18 satisfy the identity

$$X + D_i(\xi^i) = W^\alpha \frac{\delta}{\delta u^\alpha} + D_i N^i. \quad (6.20)$$

This operator identity is called the Noether identity.

With this identity in hand, the content and proof of the Noether theorem as well as related results become transparent.

For simplicity we first consider a one-parameter group  $G$  of point transformations

$$\bar{x}^i = f^i(x, u, a), \quad \bar{u}^\alpha = \varphi^\alpha(x, u, a). \quad (6.21)$$

Let

$$X = \xi^i(x, u) \frac{\partial}{\partial x^i} + \eta^\alpha(x, u) \frac{\partial}{\partial u^\alpha} \quad (6.22)$$

be its infinitesimal operator. Under the action of the group  $G$ , any function  $u(x)$  is transformed into  $\bar{u}(\bar{x})$  and a domain  $\Omega \subset \mathbb{R}^n$  is transformed into a domain  $\bar{\Omega} \subset \mathbb{R}^n$ .

**Definition 6.3.** Integral 6.14 is said to be invariant under the action of a group  $G$  of transformations 6.21 if

$$\begin{aligned} \int_{\Omega} L(x, u(x), u_1(x), \dots, u_k(x)) dx \\ = \int_{\bar{\Omega}} L(\bar{x}, \bar{u}(\bar{x}), \bar{u}_1(\bar{x}), \dots, \bar{u}_k(\bar{x})) d\bar{x} \end{aligned} \quad (6.23)$$

for every domain  $\Omega$  and for all smooth functions  $u(x)$  such that the integral exists.

**Theorem 6.2** (Ibragimov [1969], [1983]). Integral 6.14 is invariant under the action of a group  $G$  of point transformations with infinitesimal operator 6.22 iff

$$X(L) + LD_i(\xi^i) = 0. \quad (6.24)$$

The following particular case of the Noether theorem (Noether [1918]) is a direct consequence of formulas (6.20) and (6.24).

**Theorem 6.3.** If integral 6.14 is invariant under the action of a group  $G$  with infinitesimal operator 6.22 then the vector  $T = (T^1, \dots, T^n)$  defined by

$$T^i = N^i(L) \quad (6.25)$$

is a conserved vector for the Euler–Lagrange equation 6.13. In the case of an  $r$ -parameter invariance group, Formula 6.25 applies to each of the  $r$  generators of this group.

Noether [1918] assumed the existence of groups of transformations that involve derivatives, and formulated her theorem for this case. This generalization of Theorem 6.3 is what is understood today as Noether's theorem (more precisely, Noether's first theorem, as distinguished from her second theorem, which is not discussed here).

It is clear that Definition 6.3 and Theorem 6.3 can be directly extended to contact transformation groups. However, in the case of transformations involving higher derivatives, the realization of the general theorem is problematic. The problem, as discussed in Chapter 5, is in the solvability of Lie–Bäcklund equations. In order to deal properly with this general case and not lose Noether's results, it is convenient to replace the invariance of the integral 6.14 by the invariance of the differential  $n$ -form (elementary action)

$$L dx. \quad (6.26)$$

Then Theorems 6.2 and 6.3 are generalized to the following, which includes Noether's theorem.

**Noether's Theorem** (Ibragimov [1983]). Given a formal Lie–Bäcklund transformation group with the Lie–Bäcklund operator

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \cdots, \quad \xi^i, \eta^\alpha \in \mathcal{A}, \quad (6.27)$$

the differential  $n$ -form 6.26 is invariant with respect to this formal group iff

$$X(L) + LD_i(\xi^i) = 0. \quad (6.28)$$

The vector  $T = (T^1, \dots, T^n)$  defined by

$$T^i = N^i(L) \quad (6.29)$$

is a conserved vector for Equation 6.13.

**Remark.** It directly follows from the Noether identity 6.20 that if Operator 6.27 satisfies, on the frame of Equation 6.13, the divergence condition

$$X(L) + LD_i(\xi^i) = D_i(B^i), \quad B^i \in \mathcal{A}, \quad (6.30)$$

rather than the invariance condition 6.28, then the conserved vector is given by

$$T^i = N^i(L) - B^i. \quad (6.31)$$

**Example.** We illustrate the Noether theorem by applying it to Equation 6.11 with Lagrangian

$$L = \frac{m}{2} |\dot{x}|^2 - \frac{\alpha}{r}. \quad (6.32)$$

This is a one-dimensional case ( $n = 1$ ), where the independent variable is  $t$  and the three dependent variables are  $x^l$ . So, Operator 6.27 is written as

$$X = \xi \frac{\partial}{\partial t} + \eta^l \frac{\partial}{\partial x^l} + \cdots \quad (6.33)$$

Further, because Lagrangian 6.32 depends only on first derivatives, we need only consider a truncated form of Noether operator 6.18, namely,

$$N^l = \xi^l + \left( \eta^\alpha - \xi^l u_j^\alpha \right) \frac{\partial}{\partial u_i^\alpha}. \quad (6.34)$$

In our example Operator 6.34 becomes

$$N = \xi + \left( \eta^l - \xi \dot{x}^l \right) \frac{\partial}{\partial \dot{x}^l}.$$

Equation 6.11 admits a five-parameter group consisting of time translation, dilation, and rotations in space. An element of its Lie algebra, written in canonical form, is  $X = \eta^l \partial / \partial x^l$ , where

$$\eta^l = (3At + B) \dot{x}^l + (C_k^l - 2A\delta_k^l) x^k \quad (6.35)$$

with constant coefficients  $A, B, C_k^l$  such that  $C_k^l + C_l^k = 0$ . If one looks for additional symmetries linear in velocity, i.e., by replacing Equation 6.35 by

$$\eta^l = f_k^l(t, x) \dot{x}^k + h^l(t, x), \quad (6.36)$$

then the solution of the determining equation gives three additional Lie-Bäcklund operators,

$$X_{(k)} = \left( 2x^k \dot{x}^l - x^l \dot{x}^k - (x \cdot \dot{x}) \delta_k^l \right) \frac{\partial}{\partial x^l}, \quad k = 1, 2, 3. \quad (6.37)$$

In this case the divergence Equation 6.30 is satisfied, hence the conserved quantity for each operator 6.37 is given by Formula 6.31. This formula yields Laplace's conserved quantities, or equivalently the Laplace vector 6.12 (Ibragimov [1983], page 346).

For more detailed discussions of the Noether theorem and related results, applications, and generalizations, see Ibragimov [1983].

# Nonlocal Symmetry Generators via Bäcklund Transformations

The theory of Lie-Bäcklund groups gives an algorithmic method for constructing all *local symmetries* of differential equations, i.e., formal groups generated by the canonical Lie-Bäcklund operators

$$X = f(x, u, u_1, u_2, \dots, u_k) \frac{\partial}{\partial u} + \dots$$

with coordinates  $f$  belonging to the space  $\mathcal{A}$  of differential functions (analytic functions of an arbitrary finite number of local variables  $x, u, u_1, u_2, \dots$ ). However, in practice differential equations are encountered that admit an operator  $X = f \partial / \partial u + \dots$  whose coordinates  $f$  depend not only on a finite number of local variables, but also on expressions of the type involving integrals of  $u$ . Such symmetries are called *nonlocal*.

In a number of cases nonlocal symmetries may be easily obtained by a recursion operator.

**Example 1** (Ibragimov and Shabat [1979]). For the Korteweg-de Vries (KdV) equation

$$u_t = u_3 + uu_1,$$

the recursion operator has the form

$$L = D^2 + \frac{2}{3}u + \frac{1}{3}u_1 D^{-1}.$$

Acting by it on the coordinate  $f^{(1)} = 1 + tu_1$  of the canonical operator of the Galilean group, we get a new admissible operator with coordinate

$$f^{(3)} = L f^{(1)} = t(u_3 + uu_1) + \frac{1}{3}xu_1 + \frac{2}{3}u.$$

Here  $f^{(1)}, f^{(3)} \in \mathcal{A}$ , but further action of the recurrence  $L$  leads to the nonlocal symmetry operator

$$f^{(5)} = t(u_5 + \frac{5}{3}uu_3 + \frac{10}{3}u_1u_2 + \frac{5}{6}u^2u_1) + \frac{1}{3}x(u_3 + uu_1) + \frac{4}{3}u_2 + \frac{4}{9}u^2 + \frac{1}{9}u_1\varphi,$$

where  $\varphi$  is a nonlocal variable defined by the integrable system of differential equations

$$\varphi_x = u, \quad \varphi_t = u_2 + \frac{u^2}{2}.$$

This symmetry can also be found by direct calculation if one permits dependence of  $f$  on a nonlocal variable  $u_{-1}$  such that  $D(u_{-1}) = u$ . Developing this observation, one could introduce instead of  $\mathcal{A}$  the space  $\overline{\mathcal{A}}$  of analytic functions of any finite number of local  $(x, u, u_1, \dots)$  and "natural" nonlocal  $(u_{-1}, u_{-2}, \dots)$  variables and seek operators with coordinates from  $\overline{\mathcal{A}}$ . However, problems arise quickly. For example, the action of the operator  $L$  leads to nonlocal symmetry

$$\begin{aligned} f^{(7)} = & t(u_7 + \frac{7}{3}uu_5 + 7u_1u_4 + \frac{35}{3}u_2u_3 + \frac{35}{18}u^2u_3 + \frac{70}{9}uu_1u_2 + \frac{35}{18}u_1^3 + \frac{35}{54}u^3u_1) \\ & + \frac{1}{3}x(u_5 + \frac{5}{3}uu_3 + \frac{10}{3}u_1u_2 + \frac{5}{6}u^2u_1) + 2u_4 + \frac{8}{3}uu_2 + \frac{8}{27}u^3 \\ & + \frac{1}{9}(u_3 + uu_1)\varphi + \frac{1}{18}u_1\psi \end{aligned}$$

with the nonlocal variable  $\psi$  satisfying the equations

$$\psi_x = u^2, \quad \psi_t = 2uu_2 - u_1^2 + \frac{2}{3}u^3.$$

The nonlocal variable  $\psi$  found does not belong to  $\overline{\mathcal{A}}$  because in "natural" variables it is represented as an infinite sum:

$$\psi = D^{-1}(u^2) = uu_{-1} - u_1u_{-2} + u_2u_{-3} - \dots.$$

Thus, the use of "natural" nonlocal variables in the present case would be unsuccessful. The same thing is true for the symmetry

$$f^{(9)} = L f^{(7)},$$

which depends on  $\varphi$ ,  $\psi$ , and another nonlocal variable  $\theta$  defined by the system

$$\theta_x = \frac{u^3}{3} - u_1^2, \quad \theta_t = -2u_1u_2 + u_2^2 + u^2u_2 - 2uu_1^2 + \frac{u^4}{4}.$$

Generally speaking a closed theory of nonlocal symmetries can be constructed by the introduction of functions of an infinite number of variables  $u, u_1, u_{-1}, \dots$  (as is done, for example, in Kaptsov [1982]); however, here one loses the possibility of a constructive calculation of nonlocal symmetries.

Thus, for the construction of nonlocal symmetries a basic problem is the proper choice of nonlocal variables. They are defined by integrable systems of differential equations that relate the nonlocal variables to the original differential variable  $u$ . The choice of these differential equations is made on the basis of some additional considerations. In the example just considered, such an additional consideration was the principle of recursive construction of the symmetry operators.

Another choice is based on the observation (cf. Examples 2 and 3) that the existence of Bäcklund transformations can lead to extension of the admissible group due to nonlocal symmetries.

We note that the nonlocal variables obtained turn out to be connected with conservation laws (generally speaking, nonlocal ones). In Example 1 they are connected with the familiar series of conservation laws for the KdV equation; in fact, the compatibility conditions of the system of equations for  $\varphi$ ,  $\psi$ , and  $\theta$  give, respectively, the first three conservation laws of this series:

$$\begin{aligned} D_t(u) + D_x\left(-u_2 - \frac{u^2}{2}\right) &= 0, \\ D_t(u^2) + D_x\left(u_1^2 - 2uu_2 - \frac{2}{3}u^3\right) &= 0, \\ D_t\left(\frac{u^3}{3} - u_1^2\right) + D_x\left(2u_1u_3 - u_2^2 - u^2u_2 + 2uu_1^2 - \frac{u^4}{4}\right) &= 0. \end{aligned}$$

**Example 2** (Ibragimov [1983]). The equation

$$u_{tt} - u_{xx} = 0$$

is invariant with respect to the Bäcklund transformation

$$v_t - u_x = 0 \quad v_x - u_t = 0.$$

It has the nonlocal symmetry

$$f = \eta(u + v)$$

with arbitrary function  $\eta$ .

**Example 3.** The linear diffusion equation

$$u_t - u_{xx} = 0,$$

due to its invariance with respect to differentiation and integration, has nonlocal symmetries

$$f_1 = v, \quad f_2 = 2tu + w,$$

where  $v_x = u$ ,  $v_t = u_1$ , and  $w_x = xu$ ,  $w_t = xu_1 - u$ .

An approach was developed in Akhatov, Gazizov, and Ibragimov [1987c] for equations having Bäcklund transformations and used (Akhatov, Gazizov and Ibragimov [1987b], [1987d]) for constructing nonlocal symmetries of a special type, namely, quasilocal symmetries. It is important for applications that the Lie equations corresponding to the nonlocal operators can be integrated. This circumstance lets one again come to the classical problem of group classification of differential equations. Here new possibilities for the construction of group-invariant solutions are revealed.

Other approaches to nonlocal symmetries are considered in Fushchich [1979], Fushchich and Korniyak [1985], Kaptsov [1982], Khor'kova [1988], Kiso [1989], Konopel'chenko and Mokhnachev [1979], Svinolupov and Sokolov [1990], and Vinogradov and Krasil'shchik [1980], [1984].

## 7.1. QUASILOCAL SYMMETRIES (AKHATOV, GAZIZOV, AND IBRAGIMOV [1989b])

Let the systems of evolution equations

$$u_t = F, \quad F \in \mathcal{A}[x, u], \quad (7.1)$$

and

$$v_s = G, \quad G \in \mathcal{A}[y, v], \quad (7.2)$$

be connected by a Bäcklund transformation

$$\begin{aligned} s &= t, & y &= \varphi(x, u, u_1, \dots, u_k), & v^\alpha &= \Phi^\alpha(x, u, u_1, \dots, u_k), \\ \alpha &= 1, \dots, m, & \varphi, \Phi^\alpha &\in \mathcal{A}[x, u]. \end{aligned} \quad (7.3)$$

Here  $x$  and  $y$  are scalar independent variables;  $u$  and  $v$  are vector-valued differential variables with successive derivatives  $u_1, u_2, \dots, v_1, v_2, \dots$ , such as

$$D_x(u_i) = u_{i+1}, \quad u_0 = u; \quad D_y(v_i) = v_{i+1}, \quad v_0 = v.$$

Let System 7.1 admit an algebra of local symmetries, i.e., a Lie-Bäcklund algebra

$$\mathcal{A}_F[x, u] = \left\{ f_u \in \mathcal{A}[x, u] : \frac{\partial f_u}{\partial t} - \{F, f_u\} = 0 \right\},$$

where the Lie bracket  $\{F, f\}$  is defined as the  $m$ -dimensional vector from  $\mathcal{A}[x, u]$  with the components

$$\begin{aligned} \{F, f\}^\alpha &= F_*^\alpha \cdot f - f_*^\alpha \cdot F, \\ F_*^\alpha \cdot f &= \sum_{\beta=1}^m [F_*^\alpha]_\beta f^\beta, \quad [F_*^\alpha]_\beta = \sum_{i \geq 0} \frac{\partial F^\alpha}{\partial u_i^\beta} D_x^i, \quad \beta = 1, \dots, m. \end{aligned}$$

After the Bäcklund transformation given by Equations 7.3, if  $f_u \in \mathcal{A}_F[x, u]$  goes over into  $f_v \in \mathcal{A}[y, v]$ , then  $f_u$  and  $f_v$  are connected by the *transition formula* (Ibragimov [1981]),

$$D_x(\varphi) f_v^\alpha = (D_x(\varphi) \Phi_*^\alpha - D_x(\Phi^\alpha) \varphi_*) f_u. \quad (7.4)$$

The extension of Formula 7.4 to arbitrary  $f_v$  (in the general case  $f_v \notin \mathcal{A}[y, v]$ ) may lead to the introduction of new, *nonlocal*, variables  $Q^i$ , which are defined from Formula 7.4 and the corresponding conditions of tangency of the vector field  $f_v(y, v, v_1, \dots, Q)$  to the differential manifold  $[G]$ . The function  $f_v$ , calculated in this way, is a nonlocal symmetry; it is called a *quasilocal symmetry* associated with the local symmetry  $f_u$ .

**Definition.** A quasilocal symmetry  $f_v$  of Equation 7.2 associated with  $f_u \in \mathcal{A}_F[x, u]$  is a vector-valued function  $f_v$  that is connected with  $f_u$  by the transition formula 7.4 and depends not only on the local variables  $s, y, v, v_1, \dots$  but also depends on certain new variables  $Q^1, \dots, Q^l$  such that

$$\frac{\partial f_v}{\partial s} - \{G, f_v\} + \sum_{i=1}^l \frac{\partial f_v}{\partial Q^i} Q_s^i = 0. \quad (7.5)$$

In computing the bracket  $\{G, f_v\}$  we use differentiation  $D_y$  extended to the nonlocal variables  $Q^i$  in the same way as to the usual differential variables.

It is possible to use Equations 7.4 and 7.5 to obtain quasilocal symmetries  $f_u$  for Equation 7.1 associated with local symmetries  $f_v$  for Equation 7.2.

## 7.2. EXAMPLE

Let us consider the equations

$$v_t = v_x^{-4/3} v_{xx} \quad (7.6)$$

and

$$u_t = (u^{-4/3}u_x)_x, \quad (7.7)$$

connected by the Bäcklund transformation  $v_x = u$ , and translate the operator

$$X = x^2 \frac{\partial}{\partial x} - 3xu \frac{\partial}{\partial u}, \quad (7.8)$$

admitted by Equation 7.7, into the corresponding operator admitted by Equation 7.6.

In this case the transition formula has the form

$$f_u = D_x(f_v), \quad (7.9)$$

where  $f_u$  and  $f_v$  are coordinates of the canonical Lie-Bäcklund operator for Equations 7.7 and 7.6, respectively. For Operator 7.8 the coordinate  $f_u$  has the form

$$f_u = -3xu - x^2u_x.$$

Substituting this function into Equation 7.9 and using the Bäcklund transformation, we get the following equation with respect to the unknown function  $f_v$ :

$$D_x(f_v) = -3xu - x^2u_x \equiv -3xv_x - x^2v_{xx}.$$

After integration, the solution of this equation may be written in the form

$$f_v = -x^2v_x - xv + w,$$

where  $w$  is a nonlocal variable, satisfying the equation

$$D_xw = v.$$

Hence, the corresponding operator for Equation 7.6 has the form

$$Y = x^2 \frac{\partial}{\partial x} + (w - xv) \frac{\partial}{\partial v}.$$

The dependence of the nonlocal variable  $w$  on  $t$  is determined by the equation

$$D_t(w) = -3v_x^{-1/3},$$

which is the condition of invariance of Equation 7.6 with respect to the operator  $Y$ .

## *B. Body of Results*

# 8

## Ordinary Differential Equations

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### 8.1. SOME FIRST-ORDER EQUATIONS WITH A KNOWN SYMMETRY

I.  $y' = F(y).$

$$X = \frac{\partial}{\partial x}$$

$y' = F(x).$

$$X = \frac{\partial}{\partial y}$$

$y' = F(kx + ly).$

$$X = l \frac{\partial}{\partial x} - k \frac{\partial}{\partial y}$$

II.  $y' = \frac{y + xF(r)}{x - yF(r)}, r = \sqrt{x^2 + y^2}.$

$$X = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}$$

III.  $y' = F(y/x).$

$$X = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$$

**XIII.**  $xy' = y[\ln y + F(x)].$

$$X = xy \frac{\partial}{\partial y}$$

**XIV.**  $y' = P(x)y + Q(x).$

$$X = e^{\int P(x) dx} \frac{\partial}{\partial y}$$

**XV.**  $y' = P(x)y + Q(x)y^n.$

$$X = y^n e^{(1-n)\int P(x) dx} \frac{\partial}{\partial y}$$

**XVI.**  $y' = P(x)y.$

$$X = y \frac{\partial}{\partial y}$$

## 8.2. FIRST-ORDER EQUATIONS WITH SYMMETRIES LINEAR IN THE DEPENDENT VARIABLE

Consider a first-order ODE

$$y' = f(x, y). \quad (8.1)$$

The determining equation (Equation 1.43)

$$\eta_x + (\eta_y - \xi_x)f - \xi_y f^2 - \xi f_x - \eta f_y = 0$$

does not split into an overdetermined system. Therefore Equation 1.43 has an infinite set of solutions. That is why we can hope to find solutions of certain forms only. For instance, there is a sufficiently wide class of Equations 8.1 that admit operators of the form

$$X = (a(x)y + b(x)) \frac{\partial}{\partial x} + (c(x)y + d(x)) \frac{\partial}{\partial y}, \quad (8.2)$$

in other words components  $\xi$  and  $\eta$  are linear in  $y$  (see Section 8.1).

**Example 1** (Konovalov [1992b]). Let us find Operators 8.2 admitted by the equation

$$y' = f_n(x)y^n + f_p(x)y^p + f_1(x)y. \quad (8.3)$$

This equation admits an Operator 8.2 in the following cases.

**Case 1.**  $f_p = 0$  (The Bernoulli equation). The Bernoulli equation admits two ( $n \neq 2$ ) or three ( $n = 2$ ) linearly independent operators 8.2, for example,

$$X = \frac{1}{f_n} e^{(1-n)/f_1(x) dx} \left( \frac{\partial}{\partial x} + f_1 y \frac{\partial}{\partial y} \right).$$

**Case 2.**  $n, p \neq 2$  (One may consider Equation 8.3 as a perturbation of the Bernoulli equation). In this case Equation 8.3 admits an Operator 8.2 iff the function  $z = (f_p/f_n)^{1/(n-p)}$  satisfies the equation

$$z' = f_1 z + C f_n z^n \quad (8.4)$$

for some constant  $C$ .

The set of Operators 8.2 admitted by Equation 8.3 is a one-dimensional vector space spanned by

$$X = \frac{1}{f_n} z^{1-n} \frac{\partial}{\partial x} + \frac{1}{f_n} z' z^{1-n} y \frac{\partial}{\partial y}.$$

In the canonical variables  $t = \int f_n z^{n-1} dx$ ,  $u = y/z$ , Equation 8.3 is written as

$$u' = u^n - Cu + u^p.$$

**Remark.** If  $n = 3$  and  $p = 2$  (the special case of the Abel equation), then Condition 8.4 turns into a less strict condition.

**Case 3.**  $n = 2$ ,  $p = 0$  (The Riccati equation). The space of Operators 8.2 is three-dimensional, i.e.,

$$X = b \frac{\partial}{\partial x} - \left\{ \frac{(f_2 b)'}{f_2} y + \frac{1}{2f_2} \left[ \frac{(f_2 b)'}{f_2} + f_1 b \right]' \right\} \frac{\partial}{\partial y}, \quad (8.5)$$

where  $b(x)$  is any solution of the equation

$$\begin{aligned} & \left[ f_0 b + \frac{1}{2f_2} \left[ f_1 b + \frac{(f_2 b)'}{f_2} \right]' \right]' \\ &= \frac{f_1}{2f_2} \left( f_1 b + \frac{(f_2 b)'}{f_2} \right)' - f_0 \left( f_1 b + \frac{(f_2 b)'}{f_2} \right). \end{aligned}$$

The reduced Riccati equation

$$y' = y^2 + f_0 \quad (8.6)$$

admits the operator

$$X = b \frac{\partial}{\partial x} - \left( b' y + \frac{b''}{2} \right) \frac{\partial}{\partial y}, \quad (8.7)$$

where  $b$  is any solution of the equation

$$b''' + 4f_0 b' + 2f_0' b = 0. \quad (8.8)$$

In the canonical variables  $t = \int dx/b$ ,  $u = by + b'/2$ , Equation 8.6 is written as  $u' = u^2 + C$ , where the constant  $C$  is

$$C = f_0 b^2 + \frac{1}{2} b b'' - \frac{1}{4} b'^2.$$

We cannot find solutions of Equation 8.8 in the case of arbitrary function  $f_0$ , but an analysis of Equation 8.8 may be useful. The following example confirms this statement.

**Example 2** (Konovalov [1992a]). Consider the special Riccati equation,

$$y' = y^2 + a_1 x^{s_1} + \cdots + a_p x^{s_p}.$$

Let us find the solutions of Equation 8.8 in the form

$$b = A_1 x^{r_1} + \cdots + A_n x^{r_n}. \quad (8.9)$$

Here we consider only two cases,  $p = 1$  and  $p = 2$ .

$$p = 1: \quad y' = y^2 + ax^s. \quad (8.10)$$

Equation 8.10 admits a generator given by Equations 8.7 and 8.9 in the two cases (a)  $s = -2$  and (b)  $s = -2 \pm 2/(2n - 1)$ ,  $n = 1, 2, \dots$ . In the first

case,  $b = x$ . In the second case,

$$r_k^+ = \frac{2(k-1)}{2n-1}, \quad r_k^- = \frac{2(2n-k)}{2n-1}, \quad k = 1, \dots, n,$$

$$A_1 = 1, \quad A_k^\pm = (-2a)^{k-1} \frac{\prod_{m=1}^k \left( 2r_m - 2 \pm \frac{2}{2n-1} \right)}{\prod_{m=2}^k r_m(r_m-1)(r_m-2)}, \quad k = 2, \dots, n.$$

$$p = 2: \quad y' = y^2 + a_1 x^{s_1} + a_2 x^{s_2}. \quad (8.11)$$

Equation 8.11 admits a generator given by Equations 8.7 and 8.9 in the following two cases:

- $y' = y^2 + a_1 x^{-2} + a_2 x^{-2 \pm (2\sqrt{1-4a_1})/(2n-1)}$ , where  $a_1$  and  $a_2$  are arbitrary constants
- $y' = y^2 + a_1 x^{-2 \pm (1/n)} + F(a, n) x^{-2 \pm (2/n)}$ , where  $a_1$  is an arbitrary constant and  $a_2 = F(a_1, n)$  is obtained from the equation  $\det(c_{ij}) = 0$ , where  $(c_{ij})$  is a three-diagonal matrix,

$$c_{kk} = \frac{a_1}{n}(4k - 4n - 2),$$

$$c_{kk+1} = \frac{1}{n^3}(k-1)(k-1-n)(k-1-2n),$$

$$c_{kk-1} = \frac{4a_2}{n}(k-n)$$

For example, the equation  $y' = y^2 + ax^{-3/2} - 4a^2x^{-1}$  admits Operator 8.7 with  $b = 1 + 8ax^{1/2}$  ( $n = 2$ ).

### 8.3. SOME SECOND-ORDER EQUATIONS WITH A KNOWN SYMMETRY

I.  $y'' = F(y, y')$ .

$$X = \frac{\partial}{\partial x}$$

$$y'' = F(x, y').$$

$$X = \frac{\partial}{\partial y}$$

$$y'' = F(kx + ly, y').$$

$$X = l \frac{\partial}{\partial x} - k \frac{\partial}{\partial y}$$

$$\text{II. } y'' = (1 + y'^2)^{3/2} F\left(r, \frac{y - xy'}{x + yy'}\right).$$

$$X = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}$$

$$\text{III. } y'' = y'^3 F\left(y, \frac{y - xy'}{y'}\right).$$

$$X = y \frac{\partial}{\partial x}$$

$$\text{IV. } y'' + \frac{q''(y)}{q(y)} xy'^3 = y'^3 F\left(y, \frac{1}{y'} - \frac{q'(y)}{q(y)} x\right).$$

$$X = q(y) \frac{\partial}{\partial x}$$

$$\text{V. } y'' = F(x, y - xy').$$

$$X = x \frac{\partial}{\partial y}$$

$$\text{VI. } p(x)y'' - p''(x)y = F(x, p(x)y' - p'(x)y).$$

$$X = p(x) \frac{\partial}{\partial y}$$

$$\text{VII. } p^2(x)y'' + p(x)p'(x)y' = F(y, p(x)y').$$

$$X = p(x) \frac{\partial}{\partial x}$$

$$\text{VIII. } xy'' = F(y/x, y').$$

$$X = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$$

$$\text{IX. } y'' = x^{k-2} F(x^{-k}y, x^{1-k}y').$$

$$X = x \frac{\partial}{\partial x} + ky \frac{\partial}{\partial y}$$

$$\text{X. } y'' = yF(ye^{-x}, y'/y).$$

$$X = \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$$

$$\text{XI. } y'' = yF(x, y'/y).$$

$$X = y \frac{\partial}{\partial x}$$

$$\text{XII. } y'' = \frac{q'(y)}{q(y)} y'^2 + q(y)F\left(x, \frac{y'}{q(y)}\right).$$

$$X = q(y) \frac{\partial}{\partial y}$$

$$\text{XIII. } yy'' = y'^2 + y^2F(x, xy'/y - \ln y).$$

$$X = xy \frac{\partial}{\partial y}$$

$$\text{XIV. } xy'' + y' + x^2y'^3F(y, y/(xy') - \ln x).$$

$$X = xy \frac{\partial}{\partial x}$$

$$\text{XV. } x^3y'' = F(y/x, y - xy').$$

$$X = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}$$

$$\text{XVI. } x^3y'' = y'^3F(y/x, (y - xy')/y').$$

$$X = xy \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y}$$

$$\text{XVII. } x^{k+2}y'' + (1 - k)x^{k+1}y' = F(x^{-k}y, xy' - ky).$$

$$X = x^k \left( x \frac{\partial}{\partial x} + ky \frac{\partial}{\partial y} \right)$$

$$\text{XVIII. } xy^2y'' + (k - 1)xyy'^2 = y'^3F(xy^{-k}, y/y' - kx).$$

$$X = y^k \left( kx \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right)$$

## 8.4. LIE GROUP CLASSIFICATION OF SECOND-ORDER EQUATIONS

$$\text{I. } y'' = f(y, y').$$

$$G_1: \quad X_1 = \frac{\partial}{\partial x}$$

# Second-Order Partial Differential Equations with Two Independent Variables

## 9.1. LINEAR EQUATION

$$R(x, y)z_{xx} + S(x, y)z_{xy} + T(x, y)z_{yy} + P(x, y)z_x + Q(x, y)z_y + Z(x, y)z = 0$$

### 9.1.1. LIE'S CLASSIFICATION

*CLASSIFICATION (LIE CONTACT SYMMETRIES)*  
(Lie [1881a])

*Equivalence Transformations*

$$\bar{x} = f(x, y), \quad \bar{y} = g(x, y), \quad \bar{z} = \sigma(x, y)z,$$

where  $f$ ,  $g$ , and  $\sigma$  are arbitrary functions.

*Classification Result*

For arbitrary  $R$ ,  $S$ ,  $T$ ,  $P$ ,  $Q$ , and  $Z$  the symmetry Lie algebra is infinite and is spanned by

$$X_0 = z \frac{\partial}{\partial z}, \quad X_\infty = \varphi(x, y) \frac{\partial}{\partial z}$$

where  $\varphi(x, y)$  is an arbitrary solution of the equation.

The algebra extends in the following cases.

1.  $z_{xy} + Y(y)z_y + z = 0.$

$$X_1 = \frac{\partial}{\partial x}$$

2.  $z_{xy} + Q(x-y)z_y + Z(x-y)z = 0.$

$$X_1 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$$

3.  $z_{xy} + Cyz_y + z = 0.$

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y} - Cxz \frac{\partial}{\partial z}, \quad X_3 = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}$$

4.  $z_{xy} + \frac{A}{x-y}z_y + \frac{B}{(x-y)^2}z = 0.$

$$X_1 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \quad X_2 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y},$$

$$X_3 = x^2 \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y} - Axz \frac{\partial}{\partial z}$$

5.  $z_{xx} + z_y + Z(x)z = 0.$

$$X_1 = \frac{\partial}{\partial y}$$

6.  $z_{xx} + z_y = 0.$

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y},$$

$$X_3 = 2y \frac{\partial}{\partial x} + xz \frac{\partial}{\partial z}, \quad X_4 = x \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial y},$$

$$X_5 = xy \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y} + \left( \frac{1}{4}x^2 - \frac{1}{2}y \right) z \frac{\partial}{\partial z}$$

7.  $z_{xx} + z_y + (A/x^2)z = 0.$

$$X_1 = \frac{\partial}{\partial y}, \quad X_2 = x \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial y},$$

$$X_3 = xy \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y} + \left( \frac{1}{4}x^2 - \frac{1}{2}y \right) z \frac{\partial}{\partial z}$$

**CLASSIFICATION (LIE POINT SYMMETRIES)**  
(Ovsiannikov [1960])

*Equivalence Transformations*

$$\begin{aligned} 1^\circ: & \quad \bar{x} = \varphi(x), \quad \bar{y} = \psi(y), \quad \bar{z} = z; \\ 2^\circ: & \quad \bar{x} = x, \quad \bar{y} = y, \quad \bar{z} = \omega(x, y)z. \end{aligned}$$

Here  $\varphi$ ,  $\psi$ , and  $\omega$  are arbitrary functions.

*Classification Result*

For arbitrary  $A(x, y)$ ,  $B(x, y)$ , and  $C(x, y)$ , the symmetry Lie algebra is infinite and is spanned by

$$X_0 = z \frac{\partial}{\partial z}, \quad X_\infty = z_0(x, y) \frac{\partial}{\partial z},$$

where  $z_0 = z_0(x, y)$  is any solution of Equation 9.3.

The algebra extends in the following cases:

**Case 1.** If Laplace's invariants

$$h = A_x + AB - C, \quad k = B_y + AB - C \quad (9.6)$$

(they are invariants under the transformations  $2^\circ$ ) are equal to zero, i.e.,

$$h \equiv 0, \quad k \equiv 0,$$

then Equation 9.3 is equivalent to the wave equation

$$z_{xy} = 0.$$

(see Section 12.1).

**Case 2.** Let  $h \neq 0$  (or  $k \neq 0$ ). Then Equation 9.3 admits more than one additional operator iff the functions

$$p = \frac{k}{h}, \quad q = \frac{1}{h}(\log h)_{xy}$$

are constants.

If  $p$  and  $q$  are constants, then Equation 9.3 is equivalent to either the Euler–Poisson equation ( $q \neq 0$ )

$$z_{xy} - \frac{2/q}{x+y} z_x - \frac{2p/q}{x+y} z_y + \frac{4p/q^2}{(x+y)^2} z = 0 \quad (9.7)$$

*Classification Result*

For arbitrary  $H(x, y)$  the symmetry Lie algebra is infinite and is spanned by

$$X_0 = z \frac{\partial}{\partial z}, \quad X_\infty = z_0(x, y) \frac{\partial}{\partial z},$$

where  $z_0(x, y)$  is any solution of Equation 9.5.

The algebra extends in the following cases:

1.  $H = H(y)$ .

$$X_1 = \frac{\partial}{\partial x}$$

2.  $H = H(x)$ .

$$X_1 = \frac{\partial}{\partial y}$$

3.  $H = mx^{-2}$ ,  $m \neq 0$ .

$$\begin{aligned} X_1 &= \frac{\partial}{\partial y}, & X_2 &= x \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial y}, \\ X_3 &= -4xy \frac{\partial}{\partial x} - 4y^2 \frac{\partial}{\partial y} + (x^2 + 2y)z \frac{\partial}{\partial z} \end{aligned}$$

4.  $H = 0$ .

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x}, & X_2 &= \frac{\partial}{\partial y}, & X_3 &= x \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial y}, \\ X_4 &= 2y \frac{\partial}{\partial x} - xz \frac{\partial}{\partial z}, & X_5 &= xy \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y} - \left( \frac{1}{4}x^2 + \frac{1}{2}y \right) z \frac{\partial}{\partial z} \end{aligned}$$

## 9.2. CHAPLYGIN EQUATION

$$9.2.1. \quad K(\sigma)\psi_{\theta\theta} + \psi_{\sigma\sigma} = 0 \tag{9.9}$$

*LIE POINT SYMMETRIES*

(Ovsiannikov [1960])

For arbitrary  $K(\sigma)$  the symmetry Lie algebra is infinite and is spanned by

$$X_0 = \psi \frac{\partial}{\partial \psi}, \quad X_1 = \frac{\partial}{\partial \theta}, \quad X_\infty = \psi_0(\theta, \sigma) \frac{\partial}{\partial \psi},$$

where  $\psi_0(\theta, \sigma)$  is an arbitrary solution of Equation 9.9.

Equation 9.9 admits three additional operators iff the function  $K(\sigma)$  is given in the following parametric form:

$$K = \zeta^4(s), \quad \sigma = \frac{\zeta_0(s)}{\zeta(s)}, \quad (9.10)$$

where  $\zeta(s)$  and  $\zeta_0(s)$  are two linearly independent solutions of the linear second-order differential equation

$$\frac{d}{ds} \left[ \frac{1}{2} (qs^2 + C_1s + C_2) \frac{d\zeta}{ds} \right] = \zeta, \quad q = \text{const.}, \quad (9.11)$$

with  $C_1$  and  $C_2$  arbitrary constants.

The transformations

$$K(\sigma) = \frac{M}{(c\sigma + d)^4} K_0 \left( \frac{a\sigma + b}{c\sigma + d} \right), \quad (9.12)$$

$M \neq 0$ ,  $ad - bc \neq 0$ , are the equivalence transformations on the set of all functions  $K(\sigma)$  that are defined by Equations 9.10 and 9.11.

Functions  $K(\sigma)$  defined by Equations 9.10 and 9.11 and nonsimilar up to Equations 9.12 are as follows:

1.  $K(\sigma) = 1/\sigma^2$
2.  $K(\sigma) = J_0^4(\sqrt{2s})$ ,  $\sigma = Y_0(\sqrt{2s})/J_0(\sqrt{2s})$ , where  $y = J_0(x)$  and  $y = Y_0(x)$  are Bessel functions of first and second kinds, respectively, satisfying the equation

$$y'' + \frac{1}{x}y' + y = 0$$

3.  $K(\sigma) = \sigma^{-4\nu/(2\nu+1)}$ , where  $2/q = \nu(\nu + 1)$
4.  $K(\sigma) = e^\sigma$
5.  $K(\sigma) = P_\nu^4(s)$ ,  $\sigma = Q_\nu(s)/P_\nu(s)$ , where  $\zeta = P_\nu(s)$  and  $\zeta = Q_\nu(s)$  are Legendre functions of the  $n$ th degree, of first and second kinds, respectively, satisfying the equation  $[(1 - s^2)\zeta'] + \nu(\nu + 1)\zeta = 0$

$$9.2.2. \quad \varphi_\theta = -\psi_\sigma, \quad \varphi_\sigma = K(\sigma)\psi_\theta \quad (9.13)$$

#### CLASSIFICATION (LIE POINT SYMMETRIES) (Ovsiannikov [1962])

##### Equivalence Transformations

$$\theta = \frac{a}{\sqrt{c}} \bar{\theta}, \quad \sigma = a\bar{\sigma} + b, \quad \varphi = \frac{1}{\sqrt{c}} \bar{\varphi}, \quad \psi = \bar{\psi},$$

$$\bar{K}(\sigma) = cK(a\sigma + b),$$

where  $a$ ,  $b$ , and  $c$  are constants.

*Classification Result*

For arbitrary  $K(\sigma)$  the symmetry Lie algebra is infinite and is spanned by

$$X_0 = \varphi \frac{\partial}{\partial \varphi} + \psi \frac{\partial}{\partial \psi}, \quad X_1 = \frac{\partial}{\partial \theta}, \quad X_\infty = \varphi_0(\theta, \sigma) \frac{\partial}{\partial \varphi} + \psi_0(\theta, \sigma) \frac{\partial}{\partial \psi},$$

where  $\varphi_0(\theta, \sigma), \psi_0(\theta, \sigma)$  is an arbitrary solution of Equations 9.13.

System 9.13 admits two additional operators iff  $K(\sigma)$  satisfies the equation

$$\left( \frac{K}{K'} \right)'' = p \frac{K^2}{K'}, \quad p = \text{const.} \quad (9.14)$$

Additional operators have the following form:

**a.**  $p \neq 0$ ,

$$\begin{aligned} X = & -\frac{1}{p} \left[ 2 \left( \frac{K}{K'} \right)' + 1 \right] N'(\theta) \frac{\partial}{\partial \theta} + \frac{2K}{K'} N(\theta) \frac{\partial}{\partial \sigma} \\ & - \left[ \left( \frac{K}{K'} \right)' N(\theta) \varphi + \frac{K^2}{K'} N'(\theta) \psi \right] \frac{\partial}{\partial \varphi} \\ & + \left[ \frac{K}{K'} N'(\theta) \varphi - \left( \left( \frac{K}{K'} \right)' + 1 \right) N(\theta) \psi \right] \frac{\partial}{\partial \psi} \end{aligned}$$

**b.**  $p = 0$ ,

$$\begin{aligned} X = & \left[ \left( 2 \left( \frac{K}{K'} \right)' + 1 \right) \int N(\theta) d\theta - 2 \int \frac{K^2}{K'} d\sigma N'(\theta) \right] \frac{\partial}{\partial \theta} \\ & + \frac{2K}{K'} N(\theta) \frac{\partial}{\partial \sigma} - \left[ \left( \frac{K}{K'} \right)' N(\theta) \varphi + \frac{K^2}{K'} N'(\theta) \psi \right] \frac{\partial}{\partial \varphi} \\ & + \left[ \frac{K}{K'} N'(\theta) \varphi - \left( \left( \frac{K}{K'} \right)' + 1 \right) N(\theta) \psi \right] \frac{\partial}{\partial \psi}, \end{aligned}$$

where  $N(\theta)$  is a solution of the equation

$$N''(\theta) + pN(\theta) = 0.$$

The set of all solutions of Equation 9.14 contains the following nonequivalent solutions:

1.  $K = 1/\sigma^2$
2.  $K = e^{2t}$ ,  $\sigma = \int(e^{-t}/t) dt$
3.  $K = \sigma^{(1-2\nu)/\nu}$ ,  $\nu \neq 0$
4.  $K = e^\sigma$
5.  $K = ((z-1)/(z+1))^{1-2\nu}$ ,  $\sigma = \int(z-1)^{\nu-1}(z+1)^{-\nu} dz$ .

### 9.3. HOMOGENEOUS MONGE-AMPÈRE EQUATION

$$u_{xy}^2 - u_{xx}u_{yy} = 0$$

#### LIE POINT SYMMETRIES

(Chupakhin [1979], Ibragimov [1983])

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x}, & X_2 &= \frac{\partial}{\partial y}, & X_3 &= \frac{\partial}{\partial u}, & X_4 &= x \frac{\partial}{\partial x}, \\ X_5 &= y \frac{\partial}{\partial y}, & X_6 &= u \frac{\partial}{\partial u}, & X_7 &= y \frac{\partial}{\partial x}, & X_8 &= x \frac{\partial}{\partial y}, \\ X_9 &= x \frac{\partial}{\partial u}, & X_{10} &= y \frac{\partial}{\partial u}, & X_{11} &= u \frac{\partial}{\partial x}, & X_{12} &= u \frac{\partial}{\partial y}, \\ X_{13} &= x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y} + xu \frac{\partial}{\partial u}, & X_{14} &= xy \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y} + yu \frac{\partial}{\partial u}, \\ X_{15} &= xu \frac{\partial}{\partial x} + yu \frac{\partial}{\partial y} + u^2 \frac{\partial}{\partial u} \end{aligned}$$

#### LIE CONTACT SYMMETRIES

(Chupakhin [1979], Khabirov [1984])

$$X = (x\xi + y\eta + \zeta) \frac{\partial}{\partial u} + \cdots,$$

where  $\xi$ ,  $\eta$ , and  $\zeta$  are arbitrary functions of  $u_x$ ,  $u_y$ , and  $u - xu_x - yu_y$ .

**LIE-BÄCKLUND SYMMETRIES**

(Chupakhin [1979], Ibragimov [1983], Section 21.1, and Khabirov [1984])

$$X = f \frac{\partial}{\partial u} + \cdots, \quad \text{where } f = f_0 + yh + \sum_{k=1}^n Q^{k-1} x f_k,$$

where  $f_0$ ,  $h$ , and  $f_k$  are arbitrary functions of the variables  $u_x$ ,  $u_y$ ,  $\omega = u - xu_x - yu_y$ ,  $R^k u_y$ , and  $R^k \omega$  ( $k = 1, \dots, n-1$ ); differential operators  $Q$  and  $R$  are defined by the formulas

$$Q = u_{xx}^{-1/2} u_{yy}^{1/2} D_x - D_y, \quad R = u_{yy}^{1/4} u_{xx}^{-5/4} D_x \equiv u_{yy}^{-1/4} u_{xx}^{-3/4} D_y.$$

**9.4. NONHOMOGENEOUS MONGE-AMPÈRE EQUATION**

$$u_{xx} u_{yy} - u_{xy}^2 + a^2(x, y) = 0$$

**CLASSIFICATION (LIE POINT SYMMETRIES)**

(Khabirov [1990a])

*Equivalence Transformations*

$$\begin{aligned} 1^\circ: \quad & \bar{x} = b_{11}x + b_{12}y + b_1, \quad \bar{y} = b_{21}x + b_{22}y + b_2, \\ & \bar{u} = b_{11}b_{22}\alpha u + b_{31}x + b_{32}y + b_3, \quad \bar{a} = \alpha a \\ 2^\circ: \quad & \bar{x} = x(1 - \beta x - \gamma y)^{-1}, \quad \bar{y} = y(1 - \beta x - \gamma y)^{-1}, \\ & \bar{u} = u(1 - \beta x - \gamma y)^{-1}, \quad \bar{a} = a(1 - \beta x - \gamma y)^2, \end{aligned}$$

where  $b_i$ ,  $b_{ij}$ ,  $\alpha$ ,  $\beta$ , and  $\gamma$  are constants.

This 12-parametric equivalence transformation group is a projective group with the Lie algebra spanned by

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x}, & X_2 &= \frac{\partial}{\partial y}, & X_3 &= \frac{\partial}{\partial u}, & X_4 &= y \frac{\partial}{\partial x}, \\ X_5 &= x \frac{\partial}{\partial y}, & X_6 &= x \frac{\partial}{\partial u}, & X_7 &= y \frac{\partial}{\partial u}, \\ X_8 &= x \frac{\partial}{\partial x} - a \frac{\partial}{\partial a}, & X_9 &= y \frac{\partial}{\partial y} - a \frac{\partial}{\partial a}, & X_{10} &= u \frac{\partial}{\partial u} + a \frac{\partial}{\partial a}, \\ X_{11} &= x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y} + xu \frac{\partial}{\partial u} - 2ax \frac{\partial}{\partial a}, \\ X_{12} &= xy \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y} + yu \frac{\partial}{\partial u} - 2ay \frac{\partial}{\partial a}. \end{aligned}$$

*Classification Result*

For arbitrary  $a = a(x, y)$  the symmetry Lie algebra is three-dimensional and is spanned by

$$X_1 = \frac{\partial}{\partial u}, \quad X_2 = x \frac{\partial}{\partial u}, \quad X_3 = y \frac{\partial}{\partial u}.$$

The algebra extends in the following cases ( $a \neq 0$ ).

1.  $a = x^\beta \varphi(x^\alpha y)$ .

$$X_4 = x \frac{\partial}{\partial x} - \alpha y \frac{\partial}{\partial y} + (\beta - \alpha + 1)u \frac{\partial}{\partial u}$$

2.  $a = x^\beta y^\gamma$ .

$$X_4 = x \frac{\partial}{\partial x} + (\beta + 1)u \frac{\partial}{\partial u}, \quad X_5 = y \frac{\partial}{\partial y} + (\gamma + 1)u \frac{\partial}{\partial u}$$

3.  $a = x^\beta$ .

$$\begin{aligned} X_4 &= \frac{\partial}{\partial y}, & X_5 &= x \frac{\partial}{\partial y}, \\ X_6 &= y \frac{\partial}{\partial y} + u \frac{\partial}{\partial u}, & X_7 &= x \frac{\partial}{\partial x} + (\beta + 1)u \frac{\partial}{\partial u} \end{aligned}$$

4.  $a = x^{-3} y^\gamma$ .

$$\begin{aligned} X_4 &= x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y} + xu \frac{\partial}{\partial u}, & X_5 &= xy \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y} + yu \frac{\partial}{\partial u}, \\ X_6 &= x \frac{\partial}{\partial x} - 2u \frac{\partial}{\partial u}, & X_7 &= y \frac{\partial}{\partial y} + (\gamma + 1)u \frac{\partial}{\partial u} \end{aligned}$$

5.  $a = 1$ .

$$\begin{aligned} X_4 &= \frac{\partial}{\partial x}, & X_5 &= \frac{\partial}{\partial y}, & X_6 &= y \frac{\partial}{\partial x}, & X_7 &= x \frac{\partial}{\partial y}, \\ X_8 &= x \frac{\partial}{\partial x} + u \frac{\partial}{\partial u}, & X_9 &= y \frac{\partial}{\partial y} + u \frac{\partial}{\partial u} \end{aligned}$$

*Classification Result*

The Lie algebra of contact transformations is defined by the characteristic function  $W(x, y, u, u_x, u_y)$  (see Section 1.7; here we set  $v = u - xu_x - yu_y$ ).

1.  $a = 0$ .

$$W = W_0(u_x, u_y, v) + xW_1(u_x, u_y, v) + yW_2(u_x, u_y, v)$$

2.  $a = 1$ .

$$W = 2u - xu_x - yu_y + W_1(u_y - x, u_x + y) + W_2(u_y + x, u_x - y)$$

3.  $a = a(x)$  is an arbitrary function.

$$W = u - yu_y + \chi(x, u_y) + Cy, \quad \text{where } \chi_{xx} = a^2(x)\chi_{u_y u_y}$$

For each of the following functions  $a = a(x)$  there is a further extension:

i.  $a = 1/x$ .

$$W = Axu_x + u_y(xu_x - v) - 2y \ln|x|$$

ii.  $a = |x|^\beta$ ,  $\beta \neq 1, 2$ .

$$W = A[xu_x - (\beta + 1)yu_y] \\ + (\beta + 1)u_y(\beta u + 2xu_x - (2\beta + 1)yu_y) - yx^{2\beta+2}$$

iii.  $a = e^x$ .

$$W = A(u - u_x) + 2u_x u_y - 2yu_y^2 + uu_y - ye^{2x}$$

iv.  $a = (x^2 + \gamma)^{-1}$ .

$$W = xv - \gamma u_x + A[-y\bar{x} + u_y((x^2 + \gamma)u_x + xyu_y - xu)],$$

$$\bar{x} = \int \frac{dx}{x^2 + \gamma}$$

v.  $a = (x^2 + \gamma)^{-1} \exp(\delta\bar{x})$ ,  $\delta^2 + 4\gamma \neq 0$ ,  $\delta \neq 0$ .

$$W = xv - \gamma u_x + \delta yu_y \\ + A[-y\delta^{-1} \exp(2\delta\bar{x}) + u_y((\delta - 2x)u + 2(x^2 + \gamma)u_x \\ + 2y(x - \delta)u_y)],$$

$$\bar{x} = \int \frac{dx}{x^2 + \gamma}$$

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## *A. Apparatus of Group Analysis*

# Infinitesimal Calculus of Symmetry Groups

---

Basic tools needed for group analysis of differential equations are assembled here.

The main part of this chapter is classical and is discussed in well-known works, e.g., in Lie [1891], Campbell [1903], Dickson [1924], Eisenhart [1933], Ovsianikov [1962], [1978]; see also [H1]. Therefore, references are given only when the material under discussion is rather new or if it is not widely known.

## 1.1. ONE-PARAMETER GROUPS

### 1.1.1. DEFINITION

Let

$$\bar{z}^i = f^i(z, a), \quad i = 1, \dots, N, \quad (1.1)$$

be a one-parameter family of invertible transformations of points  $z = (z^1, \dots, z^N) \in \mathbb{R}^N$  into points  $\bar{z} = (\bar{z}^1, \dots, \bar{z}^N) \in \mathbb{R}^N$ . Here,  $a$  is a real parameter from a neighborhood of  $a = 0$ , and we impose the condition that Transformation 1.1 is an identity if and only if  $a = 0$ , i.e.,

$$f^i(z, 0) = z^i, \quad i = 1, \dots, N. \quad (1.2)$$

The set  $G$  of Transformations 1.1 satisfying Condition 1.2 is called a (local) one-parameter group of transformations in  $\mathbb{R}^N$  if the successive action of two transformations is identical to the action of a third transformation from  $G$ ,

i.e., if the function  $f = (f^1, \dots, f^N)$  satisfies the following group property:

$$f^i(f(z, a), b) = f^i(z, c), \quad i = 1, \dots, N, \quad (1.3)$$

where

$$c = \phi(a, b) \quad (1.4)$$

with a smooth function  $\phi(a, b)$  defined for sufficiently small  $a, b$ .

### 1.1.2. CANONICAL PARAMETER

The group parameter  $a$  is said to be canonical if the composition law 1.4 is  $\phi(a, b) = a + b$ , i.e., if the group property 1.3 has the form

$$f^i(f(z, a), b) = f^i(z, a + b), \quad i = 1, \dots, N. \quad (1.5)$$

Given an arbitrary composition law 1.4 the canonical parameter  $\bar{a}$  is defined by the formula

$$\bar{a} = \int_0^a \frac{da}{A(a)},$$

where

$$A(a) = \left. \frac{\partial \phi(a, b)}{\partial b} \right|_{b=0}.$$

**Example.** Let  $N = 1$ , and let  $\bar{z} = z + az$ . This is a one-parameter group with the composition law  $\phi(a, b) = a + b + ab$ . Here,  $A(a) = 1 + a$  and the canonical parameter is  $\bar{a} = \int_0^a da / A(a) = \ln(1 + a)$ .

We shall adopt the canonical parameter when referring to one-parameter groups.

### 1.1.3. GROUP GENERATOR (INFINITESIMAL OPERATOR)

Let  $G$  be a group of Transformations 1.1, where the functions  $f^i(z, a)$  satisfy the initial condition 1.2 and the group property 1.5. The infinitesimal transformation of the group  $G$  is

$$\bar{z}^i \simeq z^i + a \xi^i(z), \quad (1.6)$$

where

$$\xi^i(z) = \left. \frac{\partial f^i(z, a)}{\partial a} \right|_{a=0}, \quad i = 1, \dots, N. \quad (1.7)$$

The first-order linear differential operator

$$X = \xi^i(z) \frac{\partial}{\partial z^i} \quad (1.8)$$

is known as the infinitesimal operator (see, e.g., Campbell [1903]) or as the

generator of the group  $G$ ; Lie called it a symbol of the infinitesimal transformation 1.6. (see, e.g., Lie [1891], Eisenhart [1933]).

### 1.1.4. LIE EQUATIONS

Given an infinitesimal transformation 1.6, or an Operator 1.8, the Transformations 1.1 of the corresponding group  $G$  are determined by the following Lie equations with the initial data 1.2:

$$\frac{df^i}{da} = \xi^i(f), \quad f^i|_{a=0} = z^i, \quad i = 1, \dots, N. \quad (1.9)$$

### 1.1.5. THE GROUP GENERATOR IN NEW VARIABLES

Under a change of variables

$$y^i = h^i(z), \quad i = 1, \dots, N, \quad (1.10)$$

the differential operator 1.8 is transformed as follows:

$$X = X(h^i) \frac{\partial}{\partial y^i}. \quad (1.11)$$

Here  $X(h^i)$  is obtained by the action of differential operator  $X$  on the function  $h^i(z)$ ; the resulting expression

$$X(h^i) = \xi^k(z) \frac{\partial h^i(z)}{\partial z^k}$$

is written as a function of the new variables  $y$ .

### 1.1.6. CANONICAL VARIABLES

Any one-parameter group of Transformations 1.1 can be reduced, by a suitable change of variables 1.10, to the translation group, e.g., to translations along the  $y^1$  axis by the generator

$$X = \frac{\partial}{\partial y^1}. \quad (1.12)$$

Such variables  $y^i$  are referred to as canonical variables. According to Formula 1.11 a transition to canonical variables is determined by the equations

$$X(h^1) = 1, \quad X(h^2) = 0, \dots, \quad X(h^N) = 0. \quad (1.13)$$

### 1.1.7. INVARIANTS

A function  $F(z)$  is said to be an invariant of the group  $G$  if for each point  $z \in \mathbb{R}^N$  it is constant along the trajectory determined by the totality of transformed points  $\bar{z}$ :  $F(\bar{z}) = F(z)$ .

The function  $F(z)$  is an invariant of the group  $G$  with Generator 1.8 if and only if

$$X(F) \equiv \xi^i(z) \frac{\partial F}{\partial z^i} = 0. \quad (1.14)$$

Hence any one-parameter group has exactly  $N - 1$  functionally independent invariants (basis of invariants). One can take them to be the left-hand sides of  $N - 1$  first integrals  $J_1(z) = C_1, \dots, J_{N-1}(z) = C_{N-1}$  of the characteristic equations for linear partial differential equation 1.14:

$$\frac{dz^1}{\xi^1(z)} = \dots = \frac{dz^N}{\xi^N(z)}. \quad (1.15)$$

Then any other invariant is a function of  $J_1(z), \dots, J_{N-1}(z)$ .

### 1.1.8. INVARIANT EQUATIONS AND SURFACES

Let  $M$  be an  $(N - s)$ -dimensional surface in  $\mathbb{R}^N$  given by equations

$$F_k(z) = 0, \quad k = 1, \dots, s, \quad (1.16)$$

where  $F_k(z)$  are smooth functions such that

$$\text{rank} \left\| \frac{\partial F_k(z)}{\partial z^i} \right\| = s$$

at every point  $z \in M$ .

The surface  $M$  is said to be invariant with respect to the group  $G$  (or the system of Equations 1.16 admits  $G$ ) if any point  $z$  of the surface  $M$  moves along this surface under the action of  $G$ , i.e.,  $\bar{z} \in M$  if  $z \in M$ .

The surface  $M$  is invariant under the group  $G$  with the generator  $X$  if and only if

$$XF_k|_M = 0, \quad k = 1, \dots, s, \quad (1.17)$$

where the notation  $|_M$  means evaluated on  $M$ .

### 1.1.9. INVARIANT REPRESENTATION OF INVARIANT SURFACES

Let  $J_1(z), \dots, J_{N-1}(z)$  be a basis of invariants for the group  $G$ . Then equations of the form

$$\Phi_k(J_1(z), \dots, J_{N-1}(z)) = 0, \quad k = 1, \dots, s, \quad (1.18)$$

determine an invariant surface of the group  $G$  for any system of functions  $\Phi_k$ .

On the other hand, let  $M$  be a given surface invariant under the group  $G$ . We assume that the vector field  $\xi = (\xi^1, \dots, \xi^N)$  determined by Formula 1.7 is not equal to zero on  $M$ . Then the surface  $M$  can be represented by

Equations 1.16 where all functions  $F_k(z)$  are invariants of the group  $G$ , i.e., by equations of the form 1.18 with appropriately chosen functions  $\Phi_k$ .

1.1.10. FAMILIAR GROUPS

In Table 1.1, selected one-parameter groups are listed that often occur in practice.

TABLE 1.1  
One-parameter Groups on  $(x, y)$  Plane

Transformations	Generator	Invariant	Canonical Variables such that $X = \frac{\partial}{\partial t}$
Translations			
along $x$ : $\bar{x} = x + a, \bar{y} = y$	$X = \frac{\partial}{\partial x}$	$J = y$	$t = x, u = y$
along $y$ : $\bar{x} = x, \bar{y} = y + a$	$X = \frac{\partial}{\partial y}$	$J = x$	$t = y, u = x$
along $kx + ly = 0$ :			
$\bar{x} = x + la, \bar{y} = y - ka$	$X = l \frac{\partial}{\partial x} - k \frac{\partial}{\partial y}$	$J = kx + ly$	$t = x/l, u = kx + ly$
Rotation			
$\bar{x} = x \cos a + y \sin a,$ $\bar{y} = y \cos a - x \sin a$	$X = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}$	$J = x^2 + y^2$	$t = \operatorname{arctg} \frac{x}{y}, u = \sqrt{x^2 + y^2}$
Lorentz transformation			
$\bar{x} = x \operatorname{ch} a + y \operatorname{sh} a,$ $\bar{y} = y \operatorname{ch} a + x \operatorname{sh} a$	$X = y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}$	$J = y^2 - x^2$	$t = \frac{1}{2} \ln \frac{y+x}{y-x},$ $u = y^2 - x^2$
Galilean transformation			
$\bar{x} = x + ay, \bar{y} = y$	$X = y \frac{\partial}{\partial x}$	$J = y$	$t = \frac{x}{y}, u = y$
Homogeneous dilation			
$\bar{x} = xe^a, \bar{y} = ye^a$	$X = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$	$J = \frac{x}{y}$	$t = \ln x, u = \frac{x}{y}$
Nonhomogeneous dilation			
$\bar{x} = xe^a, \bar{y} = ye^{ka}$	$X = x \frac{\partial}{\partial x} + ky \frac{\partial}{\partial y}$	$J = \frac{x^k}{y}$	$t = \ln x, u = \frac{x^k}{y}$
Projective transformation			
$\bar{x} = \frac{x}{1-ax}, \bar{y} = \frac{y}{1-ax}$	$X = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}$	$J = \frac{x}{y}$	$t = -\frac{1}{x}, u = \frac{x}{y}$

## 1.2. PROLONGATION OF POINT TRANSFORMATION GROUPS

### 1.2.1. THE SPACE $\mathcal{A}$ OF DIFFERENTIAL FUNCTIONS

The universal space of modern group analysis is the set  $\mathcal{A}$  of differential functions introduced by Ibragimov [1981] (see also Ibragimov [1983], Section 19) as a generalization of differential polynomials considered by Ritt [1950]. Accordingly, the following differential algebraic notation will be used in our presentation.

The components of the vector  $z$  are taken from the following sets of different variables:

$$x = \{x^i\}, \quad u = \{u^\alpha\}, \quad u_1 = \{u_i^\alpha\}, \quad u_2 = \{u_{ij}^\alpha\}, \dots, \quad (1.19)$$

where  $\alpha = 1, \dots, m$ , and  $i, j = 1, \dots, n$ . These variables are correlated by the differentiations  $D_i$  as follows:

$$u_i^\alpha = D_i(u^\alpha), \quad u_{ij}^\alpha = D_j(u_i^\alpha) = D_j D_i(u^\alpha). \quad (1.20)$$

Hence  $u_{ij} = u_{ji}$ , and therefore  $u_2$  will contain only  $u_{ij}^\alpha$  with  $i \leq j$ .

The variables  $x$  are called independent variables, and the variables  $u$  are known as the *differential variables* with the successive derivatives  $u_1, u_2$ , etc.

A locally analytic function (i.e., locally expandable in a Taylor series with respect to all arguments) of a finite number of variables 1.19 is called a differential function. The highest order of derivatives appearing in the differential function is called the order of this function. This set of all differential functions of all finite orders is denoted by  $\mathcal{A}$ . This set is a vector space with respect to the usual addition of functions and becomes an associative algebra if multiplication is defined by the usual multiplication of functions. In addition, the space  $\mathcal{A}$  has the intrinsic property of being closed under the derivation given by the total derivatives

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \dots, \quad i = 1, \dots, n. \quad (1.21)$$

### 1.2.2. POINT TRANSFORMATION GROUPS

Let  $z = (x, u)$ . Then Transformations 1.1 are written in the form

$$\bar{x}^i = f^i(x, u, a), \quad f^i|_{a=0} = x^i, \quad (1.22)$$

$$\bar{u}^\alpha = \varphi^\alpha(x, u, a), \quad \varphi^\alpha|_{a=0} = u^\alpha. \quad (1.23)$$

A group  $G$  of transformations of this form is known as a group of point transformations in the space of dependent and independent variables.

The generator of the group  $G$  is

$$X = \xi^i(x, u) \frac{\partial}{\partial x^i} + \eta^\alpha(x, u) \frac{\partial}{\partial u^\alpha}, \quad (1.24)$$

where

$$\xi^i = \left. \frac{\partial f^i}{\partial a} \right|_{a=0}, \quad \eta^\alpha = \left. \frac{\partial \varphi^\alpha}{\partial a} \right|_{a=0}. \quad (1.25)$$

### 1.2.3. EXTENSION OF GROUP ACTIONS

Let  $\bar{D}_i$  denote differentiations in new variables  $\bar{x}^i$  given by Transformation 1.22. Then

$$D_i = D_i(f^j) \bar{D}_j \quad (1.26)$$

and Formulas 1.20 become

$$\bar{u}_i^\alpha = \bar{D}_i(\bar{u}^\alpha), \quad \bar{u}_{ij}^\alpha = \bar{D}_j(\bar{u}_i^\alpha), \dots \quad (1.20')$$

Equations 1.23, 1.26, and 1.20' yield

$$\bar{u}_j^\alpha D_i(f^j) = D_i(\varphi^\alpha), \quad (1.27)$$

or, by virtue of Formula 1.21

$$\left( \frac{\partial f^j}{\partial x^i} + u_i^\beta \frac{\partial f^j}{\partial u^\beta} \right) \bar{u}_j^\alpha = \frac{\partial \varphi^\alpha}{\partial x^i} + u_i^\beta \frac{\partial \varphi^\alpha}{\partial u^\beta}. \quad (1.27')$$

Formula 1.27 defines the extension of the group actions 1.22 and 1.23 to the first derivatives  $u$ . Transformations in  $(x, u, u)$  space given by Equations 1.22, 1.23, and 1.27<sup>1</sup> determine the one-parameter group called the first prolongation of the group  $G$  and denoted by  $G$ <sub>1</sub>. Higher-order extensions of the point transformations are obtained by successive derivations of Equation 1.27 taking into account relations 1.26 and 1.20'.

### 1.2.4. EXTENDED GENERATORS

We now write Equation 1.27 for the infinitesimal transformations 1.22 and 1.23, i.e., for  $f^i = x^i + a\xi^i$ ,  $\varphi^\alpha = u^\alpha + a\eta^\alpha$ . If we set  $\bar{u}_i^\alpha = u_i^\alpha + a\zeta_i^\alpha$  for the infinitesimal transformation of the first derivatives, it is seen from Equation 1.27 that

$$\zeta_i^\alpha = D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j). \quad (1.28)$$

Thus, given a group  $G$  with the Generator 1.24, the first prolongation  $G_1$  has the generator

$$X_1 = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha}, \quad (1.29)$$

with coordinates  $\xi^i$ ,  $\eta^\alpha$ , and  $\zeta_i^\alpha$  defined by Equations 1.25 and 1.28, respectively. The operator  $X$  extends the action of  $X$  to functions depending on  $x, u, u_1$ , and is called the first prolongation of Generator 1.24. Equation 1.28 is known as the first prolongation formula.

The second prolongation for the generator  $X$  of the group of point transformations 1.22, 1.23 is

$$X_2 = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha} + \zeta_{i_1 i_2}^\alpha \frac{\partial}{\partial u_{i_1 i_2}^\alpha}, \quad (1.30)$$

where the new coordinates  $\zeta_{i_1 i_2}^\alpha$  are defined by the following second prolongation formula:

$$\begin{aligned} \zeta_{i_1 i_2}^\alpha &= D_{i_2}(\zeta_{i_1}^\alpha) - u_{j i_1}^\alpha D_{i_2}(\xi^j) \\ &\equiv D_{i_2} D_{i_1}(\eta^\alpha) - u_j^\alpha D_{i_2} D_{i_1}(\xi^j) - u_{j i_1}^\alpha D_{i_2}(\xi^j). \end{aligned} \quad (1.31)$$

The higher order prolongation is defined recursively:

$$\zeta_{i_1 \dots i_s}^\alpha = D_{i_s}(\zeta_{i_1 \dots i_{s-1}}^\alpha) - u_{j i_1 \dots i_{s-1}}^\alpha D_{i_s}(\xi^j). \quad (1.32)$$

Introducing the quantities

$$W^\alpha = \eta^\alpha - \xi^j u_j^\alpha, \quad (1.33)$$

we can unify the prolongation formulas 1.28, 1.31, and 1.32 as follows (see, e.g., Ibragimov [1977])

$$\zeta_{i_1 \dots i_s}^\alpha = D_{i_1} \dots D_{i_s}(W^\alpha) + \xi^j u_{j i_1 \dots i_s}^\alpha, \quad s = 1, 2, \dots \quad (1.34)$$

### 1.2.5. PROLONGATION FORMULAS IN ONE DEPENDENT AND ONE INDEPENDENT VARIABLES

Let  $x$  be an independent variable, and  $y$  a dependent variable with a successive derivatives  $y', y'', \dots$ . In the present case the total derivative 1.21 is written as

$$D = \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + y'' \frac{\partial}{\partial y'} + \dots \quad (1.35)$$

The successive actions of  $D$  on a function  $h(x, y)$  yield

$$D(h) = h_x + y'h_y, \quad D^2(h) = h_{xx} + 2y'h_{xy} + y'^2h_{yy} + y''h_y, \quad (1.36)$$

where  $h_x$  and  $h_y$  denote the partial derivatives with respect to  $x$  and  $y$ .

Given the generator

$$X = \xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial u} \quad (1.37)$$

of a group of point transformation on the  $(x, y)$  plane, Equations 1.28 and 1.31 provide the following first and second prolongation formulas:

$$\zeta_1 = D(\eta) - y'D(\xi), \quad (1.38)$$

$$\zeta_2 = D(\zeta_1) - y''D(\xi) = D^2(\eta) - y'D^2(\xi) - 2y''D(\xi). \quad (1.39)$$

Use of Equations 1.36 yields

$$\zeta_1 = \eta_x + (\eta_y - \xi_x)y' - y'^2\xi_y, \quad (1.38')$$

$$\begin{aligned} \zeta_2 = & \eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 \\ & - y'^3\xi_{yy} + (\eta_y - 2\xi_x - 3y'\xi_y)y''. \end{aligned} \quad (1.39')$$

### 1.2.6. PROLONGATION FORMULAS IN ONE DEPENDENT AND TWO INDEPENDENT VARIABLES

Let  $x, y$  be independent variables, and  $u$  a dependent variable. In the present case, Derivatives 1.21 are

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xy} \frac{\partial}{\partial u_y} + \dots,$$

$$D_y = \frac{\partial}{\partial y} + u_y \frac{\partial}{\partial u} + u_{xy} \frac{\partial}{\partial u_x} + u_{yy} \frac{\partial}{\partial u_y} + \dots.$$

Operator 1.24 has the form

$$X = \xi^1(x, y, u) \frac{\partial}{\partial x} + \xi^2(x, y, u) \frac{\partial}{\partial y} + \eta(x, y, u) \frac{\partial}{\partial u}. \quad (1.40)$$

Its first and second prolongations are

$$X_1 = X + \zeta_1 \frac{\partial}{\partial u_x} + \zeta_2 \frac{\partial}{\partial u_y}, \quad (1.41)$$

$$X_2 = X_1 + \zeta_{11} \frac{\partial}{\partial u_{xx}} + \zeta_{12} \frac{\partial}{\partial u_{xy}} + \zeta_{22} \frac{\partial}{\partial u_{yy}}, \quad (1.42)$$

where

$$\zeta_1 = D_x(\eta) - u_x D_x(\xi^1) - u_y D_x(\xi^2), \quad (1.43)$$

$$\zeta_2 = D_y(\eta) - u_x D_y(\xi^1) - u_y D_y(\xi^2), \quad (1.44)$$

$$\begin{aligned} \zeta_{11} &= D_x(\zeta_1) - u_{xx} D_x(\xi^1) - u_{xy} D_x(\xi^2) \\ &= D_x^2(\eta) - u_x D_x^2(\xi^1) - u_y D_x^2(\xi^2) \\ &\quad - 2u_{xx} D_x(\xi^1) - 2u_{xy} D_x(\xi^2), \end{aligned} \quad (1.45)$$

$$\begin{aligned} \zeta_{12} &= D_y(\zeta_1) - u_{xx} D_y(\xi^1) - u_{xy} D_y(\xi^2) \\ &= D_y D_x(\eta) - u_x D_y D_x(\xi^1) - u_y D_y D_x(\xi^2) - u_{xy} D_x(\xi^1) \\ &\quad - u_{yy} D_x(\xi^2) - u_{xx} D_y(\xi^1) - u_{xy} D_y(\xi^2), \end{aligned} \quad (1.46)$$

$$\begin{aligned} \zeta_{22} &= D_y(\zeta_2) - u_{xy} D_y(\xi^1) - u_{yy} D_y(\xi^2) \\ &= D_y^2(\eta) - u_x D_y^2(\xi^1) - u_y D_y^2(\xi^2) \\ &\quad - 2u_{xy} D_y(\xi^1) - 2u_{yy} D_y(\xi^2). \end{aligned} \quad (1.47)$$

Use of the definitions of  $D_x$  and  $D_y$  yields

$$\zeta_1 = \eta_x + u_x \eta_u - u_x \xi_x^1 - (u_x)^2 \xi_u^1 - u_y \xi_x^2 - u_x u_y \xi_u^2, \quad (1.43')$$

$$\zeta_2 = \eta_y + u_y \eta_u - u_x \xi_y^1 - u_x u_y \xi_u^1 - u_y \xi_y^2 - (u_y)^2 \xi_u^2, \quad (1.44')$$

$$\begin{aligned} \zeta_{11} &= \eta_{xx} + 2u_x \eta_{xu} + u_{xx} \eta_u + (u_x)^2 \eta_{uu} - 2u_{xx} \xi_x^1 - u_x \xi_{xx}^1 - 2(u_x)^2 \xi_{xu}^1 \\ &\quad - 3u_x u_{xx} \xi_u^1 - (u_x)^3 \xi_{uu}^1 - 2u_{xy} \xi_x^2 - u_y \xi_{xx}^2 - 2u_x u_y \xi_{xu}^2 \\ &\quad - (u_y u_{xx} + 2u_x u_{xy}) \xi_u^2 - (u_x)^2 u_y \xi_{uu}^2, \end{aligned} \quad (1.45')$$

$$\begin{aligned}
 \xi_{12} = & \eta_{xy} + u_y \eta_{xu} + u_x \eta_{yu} + u_{xy} \eta_u + u_x u_y \eta_{uu} - u_{xy} (\xi_x^1 + \xi_y^2) - u_x \xi_{xy}^1 \\
 & - u_{xx} \xi_y^1 - u_x u_y (\xi_{xu}^1 + \xi_{yu}^2) - (2u_x u_{xy} + u_y u_{xx}) \xi_u^1 - (u_x)^2 \xi_{yu}^1 \\
 & - (u_x)^2 u_y \xi_{uu}^1 - u_y \xi_{xy}^2 - u_{yy} \xi_x^2 - (u_y)^2 \xi_{xu}^2 - (2u_y u_{xy} + u_x u_{yy}) \xi_u^2 \\
 & - u_x (u_y)^2 \xi_{uu}^2,
 \end{aligned} \tag{1.46'}$$

$$\begin{aligned}
 \xi_{22} = & \eta_{yy} + 2u_y \eta_{yu} + u_{yy} \eta_u + (u_y)^2 \eta_{uu} - 2u_{xy} \xi_y^1 - u_x \xi_{yy}^1 - 2u_x u_y \xi_{yu}^1 \\
 & - (u_x u_{yy} + 2u_y u_{xy}) \xi_u^1 - u_x (u_y)^2 \xi_{uu}^1 - 2u_{yy} \xi_y^2 - u_y \xi_{yy}^2 \\
 & - 2(u_y)^2 \xi_{yu}^2 - 3u_y u_{yy} \xi_u^2 - (u_y)^3 \xi_{uu}^2.
 \end{aligned} \tag{1.47'}$$

### 1.3. SYMMETRY GROUPS FOR DIFFERENTIAL EQUATIONS

#### 1.3.1. THE FRAME OF A DIFFERENTIAL EQUATION

Let  $F \in \mathcal{A}$  be a differential function of order  $p$ . Consider the equation

$$F(x, u, u_1, \dots, u_p) = 0. \tag{1.48}$$

If the variable  $u$  is considered to be a function of  $x$  so that

$$u^\alpha = u^\alpha(x), \quad u_i^\alpha = \frac{\partial u^\alpha(x)}{\partial x^i}, \dots,$$

then Equation 1.48 defines a  $p$ -order partial differential equation.

On the other hand, if  $x, u, u_1, \dots$  are treated as functionally independent variables connected only by differential relations 1.20, then Equation 1.48 determines a surface in the space of the independent variables  $x, u, u_1, \dots, u_p$ . This surface is called the frame (or skeleton) of the differential equation under consideration (Ibragimov [1992]).

The frame equation 1.48 is accompanied with its differential consequences:

$$D_i F = 0, \quad D_i D_j F = 0, \dots, \tag{1.48'}$$

where

$$D_i F = \frac{\partial F}{\partial x^i} + u_i^\alpha \frac{\partial F}{\partial u^\alpha} + \dots + u_{i_1 \dots i_p}^\alpha \frac{\partial F}{\partial u_{i_1 \dots i_p}^\alpha}.$$

The totality of points  $(x, u, u_1, \dots)$  satisfying all Equations 1.48 and 1.48' is denoted by  $[F]$ ; we shall call it the extended frame.

A system of  $p$ -order differential equations is determined by equations

$$F_k(x, u, u_1, \dots, u_p) = 0, \quad k = 1, \dots, s, \quad (1.49)$$

where  $F_k \in \mathcal{A}$ , and  $p$  is the maximum of orders of differential functions  $F_k$ . We assume that

$$\text{rank} \left\| \frac{\partial F_k}{\partial x^i}, \frac{\partial F_k}{\partial u^\alpha}, \frac{\partial F_k}{\partial u_i^\alpha}, \dots \right\| = s$$

on the frame of the differential equations under consideration. The frame is a surface defined by the system of Equations 1.49 in the space of functionally independent variables  $x, u, u_1, \dots, u_p$ . The extended frame is given by

$$F_k = 0, \quad D_i F_k = 0, \quad D_i D_j F_k = 0, \dots \quad (1.49')$$

### 1.3.2. FIRST DEFINITION OF A SYMMETRY GROUP

The system of differential equations 1.49 is said to be invariant under the group  $G$  if  $G$  converts every solution of the system under consideration into a solution of the same system. In other words, the solutions of the differential equation 1.49 are merely permuted among themselves (or are individually unaltered) by every transformation 1.22, 1.23 of the group  $G$ . Here solutions of differential equations are considered as classical ones, i.e., are assumed to be smooth functions  $u^\alpha = u^\alpha(x)$ .

If the system of Equations 1.49 is invariant under  $G$ , the group  $G$  is also known as a symmetry group for System 1.49 or a group admitted by this system.

### 1.3.3. SECOND DEFINITION AND THE INFINITESIMAL TEST FOR THE INVARIANCE OF DIFFERENTIAL EQUATIONS

The former definition is conceptually simple. But, it depends upon a knowledge of solutions of the differential equations and therefore it is mainly of theoretical value. Virtually, the following geometrical definition is utilized in the problem of finding symmetry groups.

The system of differential equations 1.49 is said to be invariant under the group  $G$  if the frame of the system is an invariant surface with respect to the  $p$ th prolongations  $G^p$  of  $G$ .

This definition does not assume the knowledge of solutions, and the invariance can be tested at once on any given differential equations via the

infinitesimal criterion given by Equations 1.17. In our case, this criterion yields

$$\left. \frac{X F_k}{p} \right|_{(1.49)} = 0, \quad k = 1, \dots, s, \quad (1.50)$$

where  $X$  is the  $p$ th prolongation for the generator  $X$  of the group  $G$ , and the notation  $\left|_{(1.49)}\right.$  means evaluated on the frame.

Equations 1.50 are called the determining equations for the generator  $X$  of a symmetry group. The determining equations can also be written in the form (summation in  $l = 1, \dots, s$ )

$$\left. \frac{X F_k}{p} \right|_{(1.49)} = \lambda_k^l F_l, \quad k = 1, \dots, s, \quad (1.50')$$

with undeterminate coefficients  $\lambda_k^l \in \mathcal{A}$ , ( $l, k = 1, \dots, s$ ) assumed to be differential functions that are bounded on the frame given by Equations 1.49.

#### 1.3.4. A SAMPLE FOR SOLUTION OF DETERMINING EQUATIONS

Consider a second-order partial differential equation

$$u_x u_{xx} + u_{yy} = 0, \quad (1.51)$$

describing a stationary transonic gas flow. Let

$$X = \xi^1 \frac{\partial}{\partial x} + \xi^2 \frac{\partial}{\partial y} + \eta \frac{\partial}{\partial u} \quad (1.52)$$

be the generator of a symmetry group we are looking for.

Use of the second prolongation formula 1.42 reduces the determining equation 1.50 to the form

$$u_{xx} \zeta_1 + u_x \zeta_{11} + \zeta_{22} = 0, \quad (1.53)$$

where we substitute  $\zeta_1$ ,  $\zeta_{11}$ , and  $\zeta_{22}$  from prolongation formulas 1.43', 1.45', and 1.47', and set  $u_{yy} = -u_x u_{xx}$ . Then we have, in Equation 1.53, the independent variables  $x$ ,  $y$ ,  $u$ ,  $u_x$ ,  $u_y$ ,  $u_{xx}$ , and  $u_{xy}$ , and the unknown functions  $\xi^1$ ,  $\xi^2$ , and  $\eta$  depending only upon  $x$ ,  $y$ , and  $u$ . Accordingly, we isolate in the left-hand side of Equation 1.53, the terms containing  $u_{xy}$ ,  $u_{xx}$ ,  $u_x$ ,  $u_y$ , and set each term equal to zero.

So, the terms containing  $u_{xy}$  are

$$-2u_{xu} \left( \xi_y^1 + u_x \xi_x^2 + u_y \xi_u^1 + (u_x)^2 \xi_u^2 \right).$$

Consequently,

$$\xi_y^1 = 0, \quad \xi_u^1 = 0, \quad \xi_x^2 = 0, \quad \xi_u^2 = 0. \quad (1.54)$$

The same argument applied to the terms containing  $u_{xx}$  yields

$$\eta_x = 0, \quad \eta_u - 3\xi_x^1 + 2\xi_y^2 = 0. \quad (1.55)$$

Then Equation 1.53 reduces to

$$\eta_{yy} = 0, \quad 2\eta_{yu} - \xi_{yy}^2 = 0. \quad (1.56)$$

Thus, Equation 1.53 splits into the overdetermined system of linear partial differential equations 1.54–1.56. The general solution of this system is

$$\xi^1 = C_1 x + C_2, \quad \xi^2 = C_3 y + C_4, \quad \eta = (3C_1 - 2C_3)u + C_5 y + C_6 \quad (1.57)$$

with six arbitrary constants  $C$ .

### 1.3.5. COMMUTATOR

Given two first-order linear differential operators 1.8,

$$X_l = \xi_l^i(z) \frac{\partial}{\partial z^i}, \quad l = 1, 2,$$

their commutator (Lie bracket) is defined to be

$$[X_1, X_2] = X_1 X_2 - X_2 X_1. \quad (1.58)$$

The commutator is again a first-order operator and can be found by the following simple formula:

$$[X_1, X_2] = (X_1(\xi_2^i) - X_2(\xi_1^i)) \frac{\partial}{\partial z^i}. \quad (1.58')$$

Let  $X_1$  and  $X_2$  be two generators of point transformation groups, i.e., two operators of the form 1.24. It follows from the prolongation formulas, that the commutator and  $p$ -order prolongation (with an arbitrary  $p$ ) obey the following rule of permutability. If

$$X = [X_1, X_2], \quad (1.59)$$

then

$$X_p = \left[ X_p^1, X_p^2 \right]. \quad (1.59')$$

### 1.3.6. THE MAIN PROPERTY OF DETERMINING EQUATIONS

Determining Equation 1.50 is a system of linear partial differential equations for functions  $\xi^i(x, u)$  and  $\eta^i(x, u)$ , and hence the set of all its solutions is a vector space. The main property of the determining equation is that the vector space of its solutions is closed under commutator, i.e., if  $X_1$  and  $X_2$  satisfy Equation 1.50 then their commutator  $X = [X_1, X_2]$  also satisfies this equation. Indeed, use of the determining equation in the form 1.50', together with Equations 1.59' and 1.58, shows that relations

$$X_p^1 F_k = \lambda_k^l F_l, \quad X_p^2 F_k = \mu_k^l F_l$$

imply

$$X_p F_k = \omega_k^l F_l,$$

where

$$\omega_k^l = X_p^1(\mu_k^l) - X_p^2(\lambda_k^l) + \mu_k^q \lambda_q^l - \lambda_k^q \mu_q^l.$$

A vector space of operators  $X$  closed under the Commutator 1.58 is called a Lie algebra of operators. Thus, the solutions of determining equations 1.50 make up a Lie algebra. Given a Lie algebra  $L$ , its dimension is defined to be the dimension of the underlying vector space. The Lie algebra  $L$  of dimension  $r$  is denoted by  $L_r$ .

For example, Solutions 1.57 of the determining equation in Section 1.3.4 make up the six-dimensional Lie algebra  $L_6$  spanned by the operators

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x}, & X_2 &= \frac{\partial}{\partial y}, & X_3 &= \frac{\partial}{\partial u}, & X_4 &= y \frac{\partial}{\partial u}, \\ X_5 &= x \frac{\partial}{\partial x} + 3u \frac{\partial}{\partial u}, & X_6 &= y \frac{\partial}{\partial y} - 2u \frac{\partial}{\partial u}. \end{aligned} \quad (1.60)$$

### 1.3.7. SYMMETRY GROUPS FOR LOCALLY SOLVABLE EQUATIONS

Consider a system of differential equations 1.49. Let  $z_0 = (x_0, u_0, \dots, u_p)$  be a generic point on the frame of this system:

$$F_k(x_0, u_0, \dots, u_p) = 0, \quad k = 1, \dots, s.$$

The system 1.49 is said to be locally solvable at  $z_0$  if there is a solution passing through this point, i.e., there exist a solution  $u = h(x)$  of differential equations 1.49 defined in a neighborhood of the point  $x_0$  such that  $u_0 = h(x_0), \dots, u_p = \partial p_h / \partial x^p(x_0)$ . The system 1.49 is said to be locally solvable if it has this property at every generic point of the frame.

Nonsingular systems of ordinary differential equations and the majority of differential equations of mathematical physics are locally solvable.

It can be shown that, for locally solvable systems, both definitions given in Sections 1.3.2 and 1.3.3 provide exactly the same symmetry group. A discussion of this equivalence is to be found in Lie [1891], Chapter 6, Section 1, Ovsiannikov [1962], Section 15.1. A recent general treatment of this subject is presented by Olver [1986], Section 2.6.

### 1.3.8. WHAT OCCURS IN PARTICULAR CASES?

If the system 1.49 is not locally solvable, it may happen that the definition given in Section 1.3.3 determines only a subgroup of the symmetry group defined in Section 1.3.2. Here is a simple illustrative example due to Olver [1986], Equations 2.117.

The system of first-order equations

$$u_x = yu, \quad u_y = 0, \quad (1.61)$$

is an overdetermined system. It is not locally solvable. Indeed, the integrability condition  $u_{xy} = u_{yx}$  yields

$$u = 0. \quad (1.61')$$

Hence, the system 1.61 is not locally solvable at points  $z_0 = (x_0, y_0, u_0, u_1)$  of the frame if  $u_0 \neq 0$ , e.g., at the point  $z_0$  with  $u_0 = 1, u_1 \equiv (u_x, u_y)_0 = (y_0, 0)$ .

Thus the set of all solutions for Equations 1.61 consists only of one function  $u = 0$ . It follows that the one-parameter group of translations along the  $y$  axis is a symmetry group of the system 1.61, according to the first definition. But it is not a symmetry group by the second definition.

### 1.3.9. THIRD DEFINITION OF A SYMMETRY GROUP

Consider the system of  $p$ -order differential equations 1.49. We denote by  $F$  the vector-valued differential function  $F = (F_1, \dots, F_s)$ , and by  $[F]$  the extended frame defined by Equations 1.49 together with their differential consequences (Equations 1.49').

Following Ibragimov [1983], Section 17.1, we introduce the third definition and the corresponding infinitesimal criterion for the invariance of differential equations:

The system of differential equations 1.49 is said to be invariant under the group  $G$  if the extended frame  $[F]$  is invariant with respect to the infinite-order prolongation of  $G$ .

The infinitesimal criterion for this invariance is written as follows (Ibragimov [1983], Theorem 17.1)

$$X F_k \Big|_{[F]} = 0, \quad k = 1, \dots, s. \quad (1.62)$$

It follows that the infinitesimal test requires only finite-order extensions both the generator  $X$  and the frame of the differential equations under consideration.

Equations 1.62 will be also called determining equations. They possess the main property of determining equations given in Section 1.3.6.

For locally solvable systems, all three definitions of symmetry groups are equivalent. For overdetermined systems, the first and third definitions are equivalent, whereas the second definition provides, in general, only a subgroup of the symmetry group given by the third definition.

### 1.3.10. ILLUSTRATION OF A DISTINCTION BETWEEN DETERMINING EQUATIONS 1.50 AND 1.62

Consider the following overdetermined system:

$$u_t = (u_x)^{-4/3} u_{xx}, \quad v_t = -3(u_x)^{-1/3}, \quad v_x = u. \quad (1.63)$$

By the notation of Section 1.3.1, this is a system of second-order differential equations with two independent variables  $t$  and  $x$ , and two dependent variables  $u$  and  $v$ . Accordingly, we take  $p = 2$  in the determining equations 1.50 and 1.62.

We first solve the determining equations 1.50. In our case, the left-hand side of Equations 1.50 depends upon the variables  $x, t, u, v, u_x, u_{xx}, u_{xt}$ , and  $v_{xx}$  in accordance with prolongation formulas 1.32 (cf. Formula 1.46'). The solution of the determining equations yields the six-dimensional Lie algebra spanned by

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x}, & X_2 &= \frac{\partial}{\partial t}, & X_3 &= \frac{\partial}{\partial v}, & X_4 &= \frac{\partial}{\partial u} + x \frac{\partial}{\partial v}, \\ X_5 &= 4t \frac{\partial}{\partial t} + 3u \frac{\partial}{\partial u} + 3v \frac{\partial}{\partial v}, & X_6 &= 2x \frac{\partial}{\partial x} - u \frac{\partial}{\partial u} + v \frac{\partial}{\partial v}. \end{aligned} \quad (1.64)$$

This is the Lie algebra of the maximal symmetry group for Equations 1.63 obtained by the second definition given in Section 1.3.3.

Consider now the determining equations 1.62. Here, we substitute  $v_{xx} = u_x$ . Hence the left-hand side of Equations 1.62 depends only upon the variables

$x, t, u, v, u_x, u_{xx}$ , and  $u_{xt}$ . It follows that the third definition (Section 1.3.9) provides a more general symmetry group than the second definition. Namely, the solution of the determining equations 1.62 yields the seven-dimensional Lie algebra spanned by Operators 1.64 and by

$$X_7 = x^2 \frac{\partial}{\partial x} + xv \frac{\partial}{\partial v} + (v - xu) \frac{\partial}{\partial u}. \quad (1.65)$$

Operator  $X_7$  generates, via Lie equations 1.9, the one-parameter group of projective transformations on the  $(x, v)$  plane

$$\bar{x} = \frac{x}{1 - ax}, \quad \bar{v} = \frac{v}{1 - ax} \quad (1.66)$$

(cf. Section 1.1.10) accompanied by the following transformation of  $u$ :

$$\bar{u} = (1 - ax)u + av. \quad (1.67)$$

## 1.4. ELEMENTARY PROPERTIES OF LIE ALGEBRAS

### 1.4.1. LIE ALGEBRA

A Lie algebra is a vector space  $L$  endowed with a bilinear product  $[X_1, X_2]$  (known as a commutator of  $X_1, X_2 \in L$ ) that is skew-symmetric:

$$[X_1, X_2] = -[X_2, X_1],$$

and satisfies the Jacobi identity:

$$[X_1, [X_2, X_3]] + [X_2, [X_3, X_1]] + [X_3, [X_1, X_2]] = 0$$

for all  $X_1, X_2, X_3 \in L$ .

In the group analysis of differential equations, we deal with real (at most complex) Lie algebras of operators (see Section 1.3.6), and hence we consider only vector spaces over the field of real (complex) numbers.

The dimension of a Lie algebra  $L$  is, by definition, the dimension of the vector space  $L$ . We shall use the symbol  $L_r$  to denote an  $r$ -dimensional Lie algebra.

### 1.4.2. THE STRUCTURE CONSTANTS

Let  $L_r$  be a vector space, and let  $X_1, \dots, X_r$  be its basis. Then  $L_r$  is a Lie algebra relative to a given commutator, i.e.,  $L_r$  is closed under this commutator, if

$$[X_\mu, X_\nu] = c_{\mu\nu}^\lambda X_\lambda$$

with constant coefficients  $c_{\mu\nu}^\lambda$  known as the structure constants.

The structure constants transform like the components of a tensor under changes of bases.

### 1.4.3. ISOMORPHISM AND AUTOMORPHISM

A linear one-to-one map  $f$  of the Lie algebra  $L$  onto the Lie algebra  $K$  is called an isomorphism (and algebras  $L$  and  $K$  are said to be isomorphic) if

$$f([X_1, X_2]_L) = [f(X_1), f(X_2)]_K,$$

where the indexes  $L$  and  $K$  are used to denote the commutator in the corresponding algebra.

Two Lie algebras are isomorphic if they have the same structure constants in an appropriately chosen basis.

An isomorphism of  $L$  onto itself is termed an automorphism.

### 1.4.4. SUBALGEBRA AND IDEAL

A vector space  $K \subseteq L$  is said to be a subalgebra of the Lie algebra  $L$  if it is closed under commutation, i.e.,  $[K, K] \subset K$ . It is an ideal of  $L$  if  $[K, L] \subset K$ , where  $[K, L]$  is the linear span of the set of all commutators  $[Y, X]$  with  $Y \in K$ , and  $X \in L$ .

### 1.4.5. QUOTIENT ALGEBRA

Let  $K$  be an ideal of  $L$ . The family  $L/K$  of pairwise disjoint cosets  $X + K$  ( $X \in L$ ) is naturally endowed with a Lie-algebraic structure. The resulting algebra  $L/K$  is called the quotient algebra of the Lie algebra  $L$  by its ideal  $K$ .

### 1.4.6. DERIVED ALGEBRAS

The Lie algebra

$$L^{(1)} = [L, L]$$

is called the derived algebra of the Lie algebra  $L$ . By construction,  $L^{(1)}$  is an ideal of  $L$ . The higher-order derived algebras are recursively defined:

$$L^{(n+1)} = (L^{(n)})^{(1)} \equiv [L^{(n)}, L^{(n)}], \quad n = 1, 2, \dots$$

A Lie algebra  $L$  is said to be abelian if  $L^{(1)} = 0$ , i.e., if all elements commute.

### 1.4.7. SOLVABLE LIE ALGEBRAS

The Lie algebra  $L_r$  is said to be solvable if there is a series

$$L_r \supset L_{r-1} \supset \cdots \supset L_1$$

of subalgebras of respective dimensions  $r, r-1, \dots, 1$  such that  $L_s$  is an ideal in  $L_{s+1}$ ,  $s = 1, \dots, r-1$ .

The Lie algebra  $L_r$  is solvable if and only if its derived algebra of a finite order vanishes:  $L_r^{(n)} = 0$ ,  $0 < n < \infty$ . It follows that any two-dimensional Lie algebra is solvable.

### 1.4.8. SIMPLE AND SEMI-SIMPLE LIE ALGEBRAS

The Lie algebra  $L$  is said to be simple if it has no ideals different from  $\{0\}$  and  $L$ . An example is the three-dimensional Lie algebra of the group of rotations in  $\mathbb{R}^3$ .

A Lie algebra is said to be semi-simple if it has no solvable ideals different from  $\{0\}$ . A Lie algebra is semi-simple if and only if it contains no abelian ideals different from  $\{0\}$ . According to Cartan's criterion, the Lie algebra  $L_r$  with the structure constants  $c_{\mu\nu}^\lambda$  is semi-simple if and only if

$$\det \|g_{\mu\nu}\| \neq 0,$$

where  $\|g_{\mu\nu}\|$  is the matrix with entries

$$g_{\mu\nu} = c_{\mu\gamma}^\lambda c_{\nu\lambda}^\gamma, \quad \mu, \nu = 1, \dots, r.$$

### 1.4.9. INNER AUTOMORPHISMS

Let  $L_r$  be an  $r$ -dimensional Lie algebra. Let a basis  $X_1, \dots, X_r$  be selected. Accordingly, the structure constants  $c_{\mu\nu}^\lambda$  are known, and any  $X \in L$  is written as

$$X = e^\mu X_\mu.$$

Hence, elements of  $L_r$  are represented by vectors  $e = (e^1, \dots, e^r)$ .

Let  $L_r^A$  be a Lie algebra spanned by the following operators:

$$E_\mu = c_{\mu\nu}^\lambda e^\nu \frac{\partial}{\partial e^\lambda}, \quad \mu = 1, \dots, r,$$

with the commutator defined by Formula 1.58. The algebra  $L_r^A$  generates (via Lie equations) the group  $G^A$  of linear transformations of  $\{e^\mu\}$ . These transformations determine automorphisms of the Lie algebra  $L_r$  known as

inner automorphisms. Accordingly,  $G^A$  is called the group of inner automorphisms of  $L_r$ , or the adjoint group of  $G$ .

#### 1.4.10. OPTIMAL SYSTEMS OF SUBALGEBRAS

Two subalgebras of  $L_r$  are said to be similar (or conjugate), if there is an inner automorphism which takes one subalgebra into the other. The similarity relation divides the set of all subalgebras of  $L_r$  into disjoint classes of conjugate subalgebras. Now in this partition we take the classes of subalgebras of the same dimension  $s$ , and choose a representative for each of the classes. The resulting set of pairwise nonconjugate subalgebras is known as an optimal system of  $s$ -dimensional subalgebras of the Lie algebra  $L_r$ .

#### 1.4.11. EXAMPLE

(Cf. [H1], Section 4.2). Consider the algebra  $L_2$  with the basis

$$X_1 = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}, \quad X_2 = x \frac{\partial}{\partial x} + \frac{y}{2} \frac{\partial}{\partial y}.$$

Here,  $[X_1, X_2] = -X_1$ , so that the nonzero structure constants are

$$c_{12}^1 = -1, \quad c_{21}^1 = 1.$$

Hence, the algebra  $L_2^A$  is spanned by

$$E_1 = -e^2 \frac{\partial}{\partial e^1}, \quad E_2 = e^1 \frac{\partial}{\partial e^1}.$$

The group  $G^A$  of inner automorphisms of  $L_2$  is a two-parameter group composed of the following two one-parameter groups of transformations:

$$\bar{e}^1 = e^1 - a_1 e^2, \quad \bar{e}^2 = e^2,$$

and

$$\bar{e}^1 = a_2 e^1, \quad \bar{e}^2 = e^2,$$

generated by  $E_1$  and  $E_2$ , respectively. It follows that any vector  $e = (e^1, e^2)$  with  $e^2 \neq 0$  is similar to  $\bar{e} = (0, e^2)$ . Hence, the optimal system of one-dimensional subalgebras is  $\{X_1, X_2\}$ .

## 1.5. MULTI-PARAMETER GROUPS

### 1.5.1. DEFINITION

Let

$$\bar{z} = f(z, a), \quad (1.68)$$

where  $z = (z^1, \dots, z^N)$  and  $a = (a^1, \dots, a^r)$ , be an  $r$ -parameter family of invertible transformations in  $\mathbb{R}^N$ . Here, the vector-function  $f = (f^1, \dots, f^N)$  is defined and sufficiently differentiable in a neighborhood of  $a = 0$ , and satisfies the initial condition:

$$f|_{a=0} = z. \quad (1.69)$$

The set of Transformations 1.68 satisfying Condition 1.69 is called a local  $r$ -parameter group  $G_r$  of transformations in  $\mathbb{R}^N$  if

$$f(f(z, a), b) = f(z, c) \quad (1.70)$$

with a composition law

$$c^\mu = \varphi^\mu(a, b), \quad \mu = 1, \dots, r, \quad (1.71)$$

where  $\varphi^\mu(a, b)$  are smooth functions defined for sufficiently small  $a$  and  $b$ .

Given  $a$ , there exists a solution  $b$  of the equations  $\varphi^\mu(a, b) = 0$ ,  $\mu = 1, \dots, r$ . This solution is unique and is denoted by  $a^{-1}$ . It provides the reconversion of Transformation 1.68 (an inverse element in the group  $G_r$ ):

$$z = f(\bar{z}, a^{-1}).$$

### 1.5.2. LIE EQUATIONS FOR A GIVEN GROUP $G_r$

Let  $G_r$  be an  $r$ -parameter group of Transformations 1.68 with the composition law 1.71. Let

$$\xi_\mu^i(z) = \left. \frac{\partial f^i(z, a)}{\partial a^\mu} \right|_{a=0}, \quad i = 1, \dots, N, \mu = 1, \dots, r, \quad (1.72)$$

and let

$$V_\mu^\nu(z) = \left. \frac{\partial \varphi^\nu(z, a)}{\partial b^\mu} \right|_{a=0}. \quad (1.73)$$

Then the functions  $f^i$  satisfy the Lie equations

$$\frac{\partial f^i}{\partial a^\mu} = V_\mu^\nu(a) \xi_\nu^i(f), \quad i = 1, \dots, N, \mu = 1, \dots, r. \quad (1.74)$$

The differential operators

$$X_\mu = \xi_\mu^i(z) \frac{\partial}{\partial z^i}, \quad \mu = 1, \dots, r, \quad (1.75)$$

with coordinates  $\xi_\mu^i(z)$  given by Formula 1.72 are generators of the group  $G_r$ ; they span  $r$ -dimensional Lie algebra called the Lie algebra of the group  $G_r$ .

### 1.5.3. CONSTRUCTION OF LIE EQUATIONS FOR A GIVEN LIE ALGEBRA $L_r$

Let  $L_r$  be an  $r$ -dimensional Lie algebra. Let its structure constants, in a fixed basis of operators of the form 1.75, are  $c_{\mu\lambda}^\nu$ . Consider the following Cauchy problem:

$$\frac{d\theta_\mu^\nu}{dt} = \delta_\mu^\nu + c_{\gamma\lambda}^\nu k^\lambda \theta_\mu^\gamma, \quad \theta_\mu^\nu|_{t=0} = 0, \quad (1.76)$$

where  $\delta_\mu^\lambda$  is the Kronecker symbol, and  $k^\lambda$  is a system of  $r$  parameters. The solution of Problem 1.76 has the form

$$\theta_\mu^\nu = th_\mu^\nu(k^1 t, \dots, k^r t).$$

Define the functions  $V_\mu^\nu(b)$  as follows:

$$V_\mu^\nu(b) = h_\mu^\nu(b^1, \dots, b^r), \quad b^\nu = k^\nu t. \quad (1.77)$$

Then Lie equations 1.74 (an overdetermined system) with these functions  $V_\mu^\nu$  are integrable. Furthermore, there exists a unique solution  $f^i(z, a)$  of the Lie equations satisfying the initial condition 1.69. This solution satisfies the group property 1.70 and hence defines an  $r$ -parameter group  $G_r$  of transformations 1.68. The Lie algebra of the group  $G_r$  is identical to the given algebra  $L_r$ .

### 1.5.4. SIMILAR GROUPS

Two  $r$ -parameter groups of transformations in  $\mathbb{R}^N$  are said to be similar if the transformations of these groups are connected by appropriate changes of the group parameters  $a = (a^1, \dots, a^r)$  and the transformed variables  $z = (z^1, \dots, z^N)$ .

Section 22 of Eisenhart [1933] provides a general algorithm, due to Lie [1888], page 356 and Eisenhart [1932], for revealing a similarity of transformation groups.

### 1.5.5. COMPOSITION OF A MULTI-PARAMETER GROUP FROM ONE-PARAMETER GROUPS WHEN A LIE ALGEBRA IS GIVEN

Let  $L_r$  be a Lie algebra, and let Operators 1.75,

$$X_\mu = \xi_\mu^i(z) \frac{\partial}{\partial z^i}, \quad \mu = 1, \dots, r,$$

constitute its basis. The construction of the corresponding  $r$ -parameter group  $G_r$  described in Section 1.5.3 is mainly of theoretical value. For applied group analysis, an easy way is to construct  $G_r$  as a composition of  $r$  one-parameter groups generated by each of the base operators  $X_\mu$  via Lie equations 1.9.

This construction depends upon the choice of a basis in  $L_r$ . However, all these  $r$ -parameter groups as well as that obtained via Lie equations from Section 1.5.3 are similar. Therefore, they are considered to be indistinguishable.

Any subalgebra  $K$  of the Lie algebra  $L_r$  generates, in a similar way, a group  $H \subset G_r$  known as a subgroup of the group  $G_r$ .

### 1.5.6. BASIS OF INVARIANTS

For multi-parameter groups, invariants are defined as in Section 1.1.7. A function  $F(z)$  is an invariant of the group  $G_r$  with the base generators 1.75 if and only if the following equations hold:

$$X_\mu F \equiv \xi_\mu^i(z) \frac{\partial F}{\partial z^i} = 0, \quad \mu = 1, \dots, r. \quad (1.78)$$

Let

$$r_* = \text{rank} \|\xi_\mu^i(z)\|, \quad (1.79)$$

the matrix  $\xi_\mu^i(z)$  being taken at generic points  $z \in \mathbb{R}^N$ . Then it follows from Equations 1.78 that  $G_r$  has  $N - r_*$  (not  $N - r$ ) functionally independent invariants  $J_1(z), \dots, J_{N-r_*}(z)$ , and any other invariant is a function of  $J_1, \dots, J_{N-r_*}$ .

Any set of  $N - r_*$  functionally independent invariants is called a basis of invariants for the group  $G_r$ .

**Example.** Consider, in  $\mathbb{R}^3$ , the group  $G_3$  of rotations with the generators

$$\begin{aligned} X_1 &= z^2 \frac{\partial}{\partial z^1} - z^1 \frac{\partial}{\partial z^2}, & X_2 &= z^3 \frac{\partial}{\partial z^2} - z^2 \frac{\partial}{\partial z^3}, \\ X_3 &= z^1 \frac{\partial}{\partial z^3} - z^3 \frac{\partial}{\partial z^1}. \end{aligned}$$

Here,  $r = 3$ , and  $r_* = 2$ . Consequently, the basis of invariants consists of one invariant, e.g., of

$$|z| = \sqrt{(z^1)^2 + (z^2)^2 + (z^3)^2}.$$

### 1.5.7. REGULAR INVARIANT SURFACES

A surface  $M \subset \mathbb{R}^N$  is said to be a regular invariant surface with respect to the group  $G_r$  if  $M$  is invariant under the action of  $G_r$  (see Definition 1.1.8) and if the following property holds:

$$r_*|_M = r_*, \quad (1.80)$$

where  $r_*$  is defined by Equation 1.79, and  $r_*|_M$  is the rank of the matrix evaluated on  $M$ .

Let  $M$  be given by Equations 1.16, and  $G_r$  be a group with Generators 1.75. Then the invariant surface criterion is written as follows (cf. Section 1.1.8):

$$X_\mu F_k|_M = 0; \quad k = 1, \dots, s; \quad \mu = 1, \dots, r. \quad (1.81)$$

Any regular invariant surface  $M$  admits an invariant representation (cf. Section 1.1.9)

$$\Phi_k(J_1(z), \dots, J_{N-r_*}(z)) = 0, \quad k = 1, \dots, s, \quad (1.82)$$

where  $J_1, \dots, J_{N-r_*}$  is a basis of invariants for the group  $G_r$ . For the proof, see Ovsianikov [1962] or [1978].

### 1.5.8. SINGULAR INVARIANT SURFACES

An invariant surface  $M$  is said to be singular (with respect to the group  $G_r$ ) if Equation 1.80 is replaced by the following inequality:

$$r_*|_M < r_*. \quad (1.83)$$

Thus, the inequality 1.83 and Equations 1.81 provide an infinitesimal test for singular invariant surfaces.

### 1.5.9. REGULAR INVARIANT DIFFERENTIAL EQUATIONS

In accordance with Sections 1.3.3 and 1.5.7, a system of  $p$ -order differential equations 1.49 is said to be a regular invariant system with respect to the group  $G_r$  of the point transformations 1.22 and 1.23 (with  $a = (a^1, \dots, a^r)$ ), if the frame of these equations is a regular invariant surface of the  $p$ th prolongation of  $G_r$ .

For a given group  $G_r$  of point transformations, the task of determining the general form of regular invariant differential equations of any order  $p$  has a simple solution. Indeed, first we find a basis of invariants for the  $p$ th prolongation of  $G_r$ . (These invariants are functions of  $x, u, u_1, \dots, u_p$  and therefore they are called  $p$ -order differential invariants.) Then we write Equations 1.82 with base differential invariants  $J_1(x, u, u_1, \dots, u_p), \dots$ , to obtain the general form of invariant equations.

**Example.** Let us find the general form of regular invariant ordinary differential equations of first and second order for the group  $G_3$  generated by

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}. \quad (1.84)$$

Here,  $x$  is the independent variable, and  $y$  is the dependent variable. The first prolongations of  $X_1, X_2, X_3$  are identical with the operators 1.84, whereas their second prolongation yields

$$X_1 = X_1, \quad X_2 = X_2, \quad X_3 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} - y'' \frac{\partial}{\partial y''}. \quad (1.84')$$

Whence, by Equation 1.79,  $r_* = 2$  for the first prolongation and  $r_* = 3$  for the second prolongation. Accordingly, the basis of the first order differential invariants consists of one invariant  $J_1 = y'$ ; the second prolongation has also only one independent invariant which is again  $J_1 = y'$ . We conclude that the general form of regular invariant first order differential equations for  $G_3$  is

$$y' = C, \quad C = \text{const}, \quad (1.85)$$

and that  $G_3$  has no regular invariant differential equations of second order.

### 1.5.10. SINGULAR INVARIANT DIFFERENTIAL EQUATIONS

A system of  $p$ -order differential equations 1.49 is said to be a singular invariant system with respect to the group  $G_r$  if the frame of these equations is a singular invariant surface of the  $p$ th prolongation of  $G_r$ .

**Example.** Consider again the group  $G_3$  with the base generators 1.84. Equation 1.79 applied to the extended operators 1.84' yields that  $r_* = 3$  at generic points  $(x, y, y', y'')$ , and that  $r_* = 2$  if and only if

$$y'' = 0. \quad (1.86)$$

It is easy to verify that the infinitesimal invariance criterion is valid for Equation 1.86 and Operators 1.84':

$$X_{\mu}^{(2)}(y'') \Big|_{y''=0} = 0; \quad \mu = 1, 2, 3.$$

Hence, Equation 1.86 is a second-order singular invariant differential equation for  $G_3$ .

### 1.5.11. REGULAR AND SINGULAR INVARIANT SOLUTIONS

Let the system of differential equations 1.49 be invariant under the group  $G$ , let  $H$  be subgroup of  $G$ . The solution

$$u^{\alpha} = h^{\alpha}(x), \quad \alpha = 1, \dots, m, \quad (1.87)$$

is said to be an  $H$ -invariant solution (for brevity, an invariant solution) if Equations 1.87 determine an invariant surface of  $H$ . This invariant surface may be regular or singular with respect to  $H$ . Accordingly, the  $H$ -invariant solution is said to be regular or singular.

The majority of exact solutions that have important real world applications, are invariant solutions. For example, the Schwarzschild metric is a singular invariant solution (with respect to the rotation group) for Einstein's equations (see Ibragimov [1983], Section 9.4).

## **Group Classification of Differential Equations Illustrated by Equations of Nonlinear Filtration**

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A variety of differential equations recognized in engineering and physical sciences as mathematical models for a diversity of natural phenomena involve arbitrary parameters or constitutive laws. Naturally, these arbitrary elements are determined experimentally or from a “simplicity criterion.” It often occurs that one can achieve the same result by requirement that arbitrary element be such that the corresponding model equation admits an additional symmetry group. Thus we come to the problem of group classification of differential equations.

The first systematic investigation of the problem of group classification was carried out by Lie [1881] for linear second-order partial differential equations with two independent variables. An English translation of this important paper is included in Part C of the present volume.

Today, there is a considerable literature on the group classification of differential equations of physical interest. These results are presented in Part B of this *Handbook*.

The purpose of this chapter is to exhibit, via a pedagogical example, the totality of practical methods available for Lie group classification of differential equations. For more details and additional examples, see Ovsianikov [1962], [1978], Akhatov, Gazizov, and Ibragimov [1989]. Classification methods in the framework of more general Lie–Bäcklund symmetries are discussed in Ibragimov [1983].

## 2.1. AN OUTLINE OF CLASSIFICATION SCHEMES

### 2.1.1. EQUIVALENCE TRANSFORMATIONS

Given a family of differential equations, an equivalence transformation is a point transformation on the  $(x, u)$  space of independent and dependent variables leaving invariant the family of equations. In other words, the equations from the family under consideration are merely permuted among themselves (or are individually unaltered) by an equivalence transformation.

The set of all equivalence transformations, for a given family of differential equations, is a group known as an equivalence group. In what follows, we denote it by  $\mathcal{G}$ .

### 2.1.2. GENERAL APPROACH TO THE PROBLEM OF GROUP CLASSIFICATION

The group classification is effected by inspecting the determining equations. An essential part of the classification is the utilization of equivalence transformations. The equivalence relation divides the set of all differential equations of a given family into disjoint classes of equivalent equations. We choose a representative for each of the classes thus simplifying the determining equations. This approach was first employed by Lie [1881]. The method, quite intricate in general, is efficient when applied to particular families of differential equations.

### 2.1.3. PARTICULAR CASE

If the group of equivalence transformations comprises all point transformations, the problem of group classification reduces to the construction of optimal systems of Lie subalgebras. Lie [1883] pursued this way in his classification of ordinary differential equations. See also [H1], Chapter 3.

### 2.1.4. PRELIMINARY CLASSIFICATION

One can observe in applications of the group analysis that most of symmetry groups resulted from the group classification are, in effect, subgroups of equivalence groups. Accordingly, one can consider the restricted problem of group classification by taking symmetry groups from equivalence transformations only. Then the problem reduces to the particular case described in the previous section. This approach was employed by Akhatov, Gazizov, and Ibragimov [1989], Sections 11 and 12, and was called a method of preliminary group classification. For more discussions and new applications, see Ibragimov, Torrisi, and Valenti [1991], Ibragimov and Torrisi [1992], and Harin [1993].

## 2.2. PEDAGOGICAL EXAMPLE

(See [H1], Section 10.3.)

Consider the family of differential equations

$$u_t = h(u_x)u_{xx} \quad (2.1)$$

describing the motion of a non-Newtonian, weakly compressible fluid in a porous medium with a nonlinear filtration law  $h(u_x)$ ,  $h' \neq 0$ .

### 2.2.1. EQUIVALENCE GROUP $\mathcal{G}$ AND ITS LIE ALGEBRA $L_{\mathcal{G}}$

According to Section 2.1.1, an equivalence transformation is a nondegenerate change of variables  $t, x, u$  taking any Equation 2.1 with an arbitrary function  $h$  into an equation of the form 2.1, generally speaking, with a different function  $h$ .

To find the continuous group of equivalence transformations, we seek its generator

$$Y = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u} + \zeta_1 \frac{\partial}{\partial u_t} + \zeta_2 \frac{\partial}{\partial u_x} + \mu \frac{\partial}{\partial h} \quad (2.2)$$

from the condition of invariance for the family of Equations 2.1 written as the system

$$u_t = hu_{xx}, \quad h_t = 0, \quad h_x = 0, \quad h_u = 0, \quad h_{u_t} = 0. \quad (2.3)$$

Here,  $u$  and  $h$  are considered as differential variables (see Section 1.2.1):  $u$  over the space of independent variables  $(t, x)$ , and  $h$  over the space of independent variables  $(t, x, u, u_t, u_x)$ . Accordingly, the coordinates  $\xi^1$ ,  $\xi^2$ , and  $\eta$  of  $Y$  are sought as functions of  $t, x, u$  (so that  $\zeta_1$  and  $\zeta_2$  are given by prolongation formulas 1.29), whereas the coordinate  $\mu$  is sought as a function of  $t, x, u, u_t, u_x, h$ .

By solving the corresponding determining equations (i.e., the infinitesimal test for invariance of the system 2.3), we obtain (detailed calculations are to be found in Akhatov, Gazizov, and Ibragimov [1989], Section 3.1) the equivalence Lie algebra  $L_{\mathcal{G}}$  spanned by

$$\begin{aligned} Y_1 &= \frac{\partial}{\partial t}, & Y_2 &= \frac{\partial}{\partial x}, & Y_3 &= \frac{\partial}{\partial u}, & Y_4 &= t \frac{\partial}{\partial t} - u_t \frac{\partial}{\partial u_t} - h \frac{\partial}{\partial h}, \\ Y_5 &= t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + u \frac{\partial}{\partial u} + h \frac{\partial}{\partial h}, & Y_6 &= u \frac{\partial}{\partial u} + u_t \frac{\partial}{\partial u_t} + u_x \frac{\partial}{\partial u_x}, \\ Y_7 &= x \frac{\partial}{\partial u} + \frac{\partial}{\partial u_x}, & Y_8 &= u \frac{\partial}{\partial x} - u_x u_t \frac{\partial}{\partial u_t} - u_x^2 \frac{\partial}{\partial u_x} + 2u_x h \frac{\partial}{\partial h}. \end{aligned} \quad (2.4)$$

It is easy to see that the reflections

$$t \Rightarrow -t \quad \text{and} \quad x \Rightarrow -x \quad (2.5)$$

yield also equivalence transformations.

Operators 2.4 and Reflections 2.5 generate the general equivalence group  $\mathcal{G}$  for Equations 2.1.

### 2.2.2. DETERMINING EQUATIONS

The generator of a group admitted by Equation 2.1, is sought in the form (cf. Operator 2.2)

$$X = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u}. \quad (2.6)$$

Consider the determining equation

$$\zeta_1 - h'(u_x)u_{xx}\zeta_2 - h(u_x)\zeta_{22} = 0,$$

where  $\zeta_1, \zeta_2, \zeta_{22}$  are calculated according to prolongation formulas 1.29 and 1.32, and  $u_t$  is replaced by  $h(u_x)u_{xx}$ . We set coefficients of  $u_{xx}$  and  $u_{tx}$  equal to zero and split the determining equation into the following system (see Section 1.3.4):

$$\xi^1 = \xi^1(t), \quad (2.7)$$

$$(2\xi_x^2 - \xi_t^1 + 2\xi_u^2 u_x)h = [\eta_x + (\eta_u - \xi_x^2)u_x - \xi_u^2(u_x)^2]h', \quad (2.8)$$

$$\begin{aligned} \eta_t - \xi_t^2 u_x = & [\eta_{xx} + 2(\eta_{xu} - \xi_{xx}^2)u_x \\ & + (\eta_{uu} - 2\xi_{xu}^2)(u_x)^2 - \xi_{uu}^2(u_x)^3]h. \end{aligned} \quad (2.9)$$

### 2.2.3. THE PRINCIPAL LIE ALGEBRA $L_{\mathcal{P}}$

In the case of an arbitrary function  $h(u_x)$  Equations 2.8 and 2.9 split into the system

$$\begin{aligned} \xi_t^1 - 2\xi_x^2 &= 0, & \xi_u^2 &= 0, & \eta_x &= 0, & \eta_u - \xi_x^2 &= 0, \\ \eta_t &= 0, & \xi_t^2 &= 0, & \xi_{xx}^2 &= 0, & \eta_{uu} &= 0. \end{aligned}$$

This system, together with Equations 2.7, yields

$$\xi^1 = C_1 + 2C_4t, \quad \xi^2 = C_2 + C_4x, \quad \eta = C_3 + C_4u. \quad (2.10)$$

Hence, in the case of an arbitrary function  $h(u_x)$ , Equation 2.1 admits the Lie algebra spanned by

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial u}, \quad X_4 = 2t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + u \frac{\partial}{\partial u}. \quad (2.11)$$

This algebra is called the principal Lie algebra  $L_{\mathcal{P}}$  for Equations 2.1.

Thus, the problem of group classification reduces to determining, from Equations 2.8 and 2.9, all particular functions  $h(u_x)$  when an extension of the principal Lie algebra  $L_{\mathcal{P}}$  occurs.

#### 2.2.4. CLASSIFYING RELATION

It follows from Equation 2.8 that the function  $h(u_x)$  satisfies the equation

$$(a + 2bu_x)h = (c + du_x - b(u_x)^2)h' \quad (2.12)$$

with constant coefficients  $a$ ,  $b$ ,  $c$ , and  $d$ . Indeed, the function  $h$  depends only on  $u_x$ . Therefore, it is only possible for Equation 2.8 to hold when its coefficients either vanish identically or are proportional to a function  $\lambda(t, x, u)$ , i.e.,

$$2\xi_x^2 - \xi_t^1 = a\lambda, \quad \xi_u^2 = b\lambda, \quad \eta_x = c\lambda, \quad \eta_u - \xi_x^2 = d\lambda.$$

If  $b = c = d = 0$  then  $a = 0$ , and we come back to Equations 2.10. Consequently, an extension of the principal Lie algebra is possible only for the functions  $h$  satisfying Equation 2.12 such that  $a + 2bu_x \neq 0$ ,  $c + du_x - bu_x^2 \neq 0$ . Then we rewrite Equation 2.12 in the form

$$\frac{h'}{h} = \frac{a + 2bu_x}{c + du_x - b(u_x)^2} \quad (2.13)$$

and call it the classifying relation.

#### 2.2.5. SIMPLIFICATION AND SOLUTION OF THE CLASSIFYING RELATION

We first simplify Equation 2.13 by means of equivalence transformations. We find that Equations 2.13 can be transformed into the following three canonical forms in accordance with whether  $\Delta = 0$ ,  $\Delta > 0$ , or  $\Delta < 0$ , where

$$\Delta = d^2 + 4bc:$$

$$\frac{h'}{h} = 1, \quad \text{if } \Delta = 0, \quad (2.14)$$

$$\frac{h'}{h} = \frac{\sigma - 1}{u_x}, \quad \sigma \geq 0, \text{ if } \Delta > 0, \quad (2.15)$$

$$\frac{h'}{h} = \frac{\sigma - 2u_x}{1 + u_x^2}, \quad \sigma \geq 0, \text{ if } \Delta < 0. \quad (2.16)$$

The integration of Equations 2.14–2.16 provides the following representatives of all functions  $h(u_x)$  when an extension of the principal Lie algebra  $L_{\mathcal{P}}$  is possible:

$$h = \exp(u_x), \quad (2.17)$$

$$h = (u_x)^{\sigma-1}, \quad \sigma \geq 0, \sigma \neq 1, \quad (2.18)$$

$$h = (1 + u_x^2)^{-1} \exp(\sigma \operatorname{arctg} u_x), \quad \sigma \geq 0. \quad (2.19)$$

## 2.2.6. THE RESULT OF THE CLASSIFICATION

Now we solve the determining equations with functions  $h$  given by Equations 2.17, 2.18, and 2.19. It follows that, for each case, the symmetry Lie algebra is five-dimensional and is spanned by Operators 2.11 and  $X_5$ , where

$$X_5 = t \frac{\partial}{\partial t} - x \frac{\partial}{\partial u} \quad \text{for } u_t = \exp(u_x) u_{xx}, \quad (2.20)$$

$$X_5 = (1 - \sigma) t \frac{\partial}{\partial t} + u \frac{\partial}{\partial u} \quad \text{for } u_t = (u_x)^{\sigma-1} u_{xx}, \quad \sigma \geq 0, \sigma \neq 1, \quad (2.21)$$

$$X_5 = \sigma t \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} - x \frac{\partial}{\partial u} \quad \text{for } u_t = (1 + u_x^2)^{-1} \exp(\sigma \operatorname{arctg} u_x) u_{xx}, \quad \sigma \geq 0. \quad (2.22)$$

## 2.2.7. UTILIZATION OF THE METHOD OF PRELIMINARY GROUP CLASSIFICATION

We define two projections,  $\pi_1$  and  $\pi_2$ , of the algebra  $L_{\mathcal{P}}$  as follows. We restrict the action of equivalence operators 2.2 to the  $(t, x, u)$  and  $(u_x, h)$

spaces, respectively, by setting

$$X = \pi_1(Y) = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u}, \quad (2.23)$$

$$Z = \pi_2(Y) = \zeta_2 \frac{\partial}{\partial u_x} + \mu \frac{\partial}{\partial h}. \quad (2.24)$$

These projections are well defined. Indeed, the coordinates  $\xi^1$ ,  $\xi^2$ , and  $\eta$  of  $Y$  are functions of  $t, x, u$  by definition, and the coordinates  $\zeta_2$  and  $\mu$  depend only upon the variables  $u_x, h$  according to Equations 2.4. Hence, we obtain the Lie algebras

$$L^1_{\mathcal{G}} = \pi_1(L_{\mathcal{G}}), \quad L^2_{\mathcal{G}} = \pi_2(L_{\mathcal{G}}), \quad (2.25)$$

of operators  $X$  and  $Z$ , respectively. Any subalgebra  $L \subset L_{\mathcal{G}}$  is projected onto the subalgebras  $\pi_1(L) = L^1 \subset L^1_{\mathcal{G}}$  and  $\pi_2(L) = L^2 \subset L^2_{\mathcal{G}}$ .

The method of preliminary group classification is based on the following statement (cf. Ibragimov and Torrisi [1992]).

Let  $L$  be a subalgebra of the equivalence algebra  $L_{\mathcal{G}}$ . Then

$$L^1 = \pi_1(L) \quad (2.26)$$

is a symmetry algebra for Equation 2.1 with the filtration law

$$h = h(u_x) \quad (2.27)$$

if and only if Equation 2.27 is invariant under the group of transformations in the  $(u_x, h)$  space generated by the Lie algebra

$$L^2 = \pi_2(L). \quad (2.28)$$

Let a subalgebra  $\bar{L}^2 \subset L^2_{\mathcal{G}}$  be similar to  $L^2$  under transformations of the equivalence group  $\mathcal{G}$ , and let  $h = \bar{h}(u_x)$  be an invariant equation with respect to  $\bar{L}^2$ . Then Equations 2.1 with the filtration laws  $h = h(u_x)$  and  $h = \bar{h}(u_x)$  are equivalent.

Thus, the problem of preliminary group classification of Equations 2.1 reduces to the algebraic problem of constructing optimal systems of subalgebras in  $L^2_{\mathcal{G}}$ . Moreover, it can be shown that invariant Equations 2.27 exist only for one-dimensional subalgebras. Hence, we have to solve the simplest problem of constructing an optimal system of one-dimensional subalgebras.

Projection 2.24 applied to base operators 2.4 of  $L_{\mathcal{G}}$  yields the following basis for the Lie algebra  $L^2_{\mathcal{G}}$ :

$$Z_1 = \frac{\partial}{\partial u_x}, \quad Z_2 = h \frac{\partial}{\partial h}, \quad Z_3 = u_x \frac{\partial}{\partial u_x}, \quad Z_4 = u_x^2 \frac{\partial}{\partial u_x} - 2u_x h \frac{\partial}{\partial h}, \quad (2.29)$$

where

$$Z_1 = \pi_2(Y_7), \quad Z_2 = \pi_2(-Y_4), \quad Z_3 = \pi_2(Y_6), \quad Z_4 = \pi_2(-Y_8). \quad (2.30)$$

Following Sections 1.4.9–1.4.11 and using Reflections 2.5, one can find the optimal system of one-dimensional subalgebras of  $L_g^2$ :

$$Z_1 + Z_2, \quad Z_3 + (\sigma - 1)Z_2, \quad Z_1 + Z_4 + \sigma Z_2, \quad Z_1 + Z_3, \quad Z_2, \quad (2.31)$$

where  $\sigma$  is a non-negative parameter. That  $\sigma \geq 0$ , in  $Z_3 + (\sigma - 1)Z_2$ , is achieved by the equivalence transformation

$$\bar{t} = t, \quad \bar{x} = u, \quad \bar{u} = x, \quad \bar{u}_x = (u_x)^{-1}, \quad \bar{h} = (u_x)^2 h.$$

Now we find the invariant equations 2.27 for each subalgebra of the optimal system. For the operator

$$Z_1 + Z_2 = \frac{\partial}{\partial u_x} + h \frac{\partial}{\partial h},$$

the invariance criterion for Equation 2.27 has the form (see Section 1.1.8, Equation 1.17):

$$(Z_1 + Z_2)(h - h(u_x))|_{h=h(u_x)} \equiv h(u_x) - h'(u_x) = 0,$$

whence, on integration (setting the constant of integration equal to 1),

$$h(u_x) = \exp(u_x).$$

Thus, the invariance with respect to the first operator of the optimal system 2.31 yields the filtration law given by Equation 2.17. Use of Relations 2.30 shows that

$$Z_1 + Z_2 = \pi_2(Y_7 - Y_4),$$

whence Equation 2.23, together with Equations 2.4, yield the additional symmetry operator (cf. Equation 2.20):

$$X_5 = \pi_1(Y_7 - Y_4) = -t \frac{\partial}{\partial t} + x \frac{\partial}{\partial u}.$$

Use of the second and third operators from the optimal system 2.31 yields the Equations 2.18 and 2.19, respectively.

Use of  $Z_1 + Z_3$  and  $Z_2$  yield  $h = \text{const}$  and  $h = 0$ . We have eliminated these cases.

Thus, we have accomplished the group classification given in Section 2.2.6, by the simple method of preliminary classification.

# Invariance Principle in Linear Second-Order Partial Differential Equations

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Here, the main emphasis is on applications of Lie group theoretic philosophy to the initial value problems.

## 3.1. EQUATIONS WITH NONDEGENERATE PRINCIPAL PART

Consider the linear second-order partial differential equation in  $n$  independent variables  $x^i$ , and one dependent variable  $u$ ,

$$F[u] \equiv a^{ij}(x)u_{ij} + b^i(x)u_i + c(x)u = 0. \quad (3.1)$$

The coefficients  $a^{ij}$ ,  $b^i$ , and  $c$  all influence the nature of the differential equation. However, the major role falls to the coefficients  $a^{ij}$  of the second derivatives (the principal part of the equation). We assume, in this section, that the principal part of Equation 3.1 is nondegenerate, i.e.,

$$\det\|a^{ij}\| \neq 0. \quad (3.2)$$

In this case, the geometric foundations of the theory are more elaborate than for equations of the general form. An excellent general account of this background based on Riemannian geometry, is to be found in Hadamard [1923], Duff [1956], and Ovsiannikov [1962].

### 3.1.1. THE COVARIANT FORM OF THE DIFFERENTIAL EQUATION

The Riemannian space  $V_n$  associated with Equation 3.1 is defined by the metric

$$ds^2 = g_{ij}(x) dx^i dx^j,$$

where  $\|g_{ij}(x)\|$  is the inverse matrix for  $\|a^{ij}(x)\|$ .

Equation 3.1 is rewritten in the following covariant form:

$$a^{ij}u_{,ij} + a^i u_i + cu = 0, \quad (3.3)$$

where

$$u_{,ij} = u_{ij} - u_k \Gamma_{ij}^k$$

is the second contravariant derivative in  $V_n$ , and  $\Gamma_{ij}^k$  denote Christoffel symbols of  $V_n$ .

### 3.1.2. EQUIVALENCE TRANSFORMATIONS AND THEIR INVARIANTS

(Cotton [1900], Ovsianikov [1962], Chapter 6; see also Ovsianikov [1978], Section 27, and Ibragimov [1983], Section 10.)

For Equations 3.1, the group of equivalence transformations comprises coordinate changes in  $V_n$ :

$$x'^i = f^i(x), \quad (3.4)$$

a linear substitution of the dependent variable:

$$u' = \alpha(x)u, \quad \alpha(x) \neq 0, \quad (3.5)$$

and a simultaneous gauge transformation of the coefficients of  $F$ , i.e.,

$$F' = \beta(x)F, \quad \beta(x) \neq 0. \quad (3.6)$$

Transformations 3.5 and 3.6 map the associated Riemannian space  $V_n$  onto conformal spaces. It is convenient to deal with the following, space preserving, combination of these transformations:

$$F'[u] = e^{-\varphi(x)} F[ue^{\varphi(x)}]. \quad (3.7)$$

After Transformation 3.7, the coefficients of Equation 3.3. become

$$a'^{ij} = a^{ij}, \quad a'^i = a^i + 2a^{ij}\varphi_j, \quad c' = c + a^{ij}(\varphi_{,ij} + \varphi_i \varphi_j) + a^i \varphi_i.$$

Hence, the following quantities remain unaltered under Transformations 3.7:

$$K_{ij} = (g_{il}a^l)_{,j} - (g_{jl}a^l)_{,i}, \quad i, j = 1, \dots, n, \quad (3.8)$$

$$H = -2c + a^i_{,ii} + \frac{1}{2}g_{ij}a^ia^j + \frac{n-2}{2(n-1)}R, \quad (3.9)$$

where a lower index following a comma denotes the covariant differentiation, and  $R$  is the scalar curvature of  $V_n$ . These quantities are Cotton's differential invariants (Cotton [1990]).

### 3.1.3. DETERMINING EQUATIONS

The symmetry Lie algebra, for any Equation 3.1, contains an ideal spanned by

$$X_0 = u \frac{\partial}{\partial u} \quad \text{and} \quad X_\tau = \tau(x) \frac{\partial}{\partial u}, \quad (3.10)$$

where  $\tau(x)$  solves Equation 3.1. Taking the quotient algebra by this ideal, one can write infinitesimal operators, admitted by Equation 3.1, in the form

$$X = \xi^i(x) \frac{\partial}{\partial x^i} + \sigma(x)u \frac{\partial}{\partial u}, \quad (3.11)$$

where the function  $\sigma(x)$  is determined up to an additive constant.

The determining equations, for Operators 3.11 admitted by Equation 3.3, are written in the following invariant form (Ovsiannikov [1962], [1978]):

$$\xi_{i,j} + \xi_{j,i} = \mu g_{ij}, \quad (3.12)$$

$$\sigma_{,i} = \frac{2-n}{4}\mu_{,i} - \frac{1}{2}(a_j\xi^j)_{,i} - \frac{1}{2}K_{ij}\xi^j, \quad (3.13)$$

$$(K_{il}\xi^l)_{,j} - (K_{jl}\xi^l)_{,i} = 0, \quad (3.14)$$

$$\xi^i H_{,i} + \mu H = 0. \quad (3.15)$$

Here

$$\xi_i = g_{ij}\xi^j, \quad a_i = g_{ij}a^j. \quad (3.16)$$

### 3.1.4. EQUATIONS WITH TWO INDEPENDENT VARIABLES

For hyperbolic equations with two independent variables,

$$u_{xy} + a(x, y)u_x + b(x, y)u_y + c(x, y)u = 0, \quad (3.17)$$

quantities  $K_{ij}$  and  $H$  are reduced to the Laplace invariants (Laplace [1773]), Darboux [1915])

$$h = a_x + ab - c, \quad k = b_y + ab - c. \quad (3.18)$$

Namely

$$K_{12} = 2(k - h), \quad H = h + k.$$

For elliptic equations

$$u_{xx} + u_{yy} + a(x, y)u_x + b(x, y)u_y + c(x, y)u = 0, \quad (3.19)$$

the invariants have the form

$$K_{12} = a_y - b_x, \quad H = -2c + a_x + b_y + \frac{1}{2}(a^2 + b^2). \quad (3.20)$$

For Equation 3.17, a symmetry generator has the form

$$X = \xi(x) \frac{\partial}{\partial x} + \eta(y) \frac{\partial}{\partial y} + \sigma(x, y)u \frac{\partial}{\partial u}, \quad (3.21)$$

and is determined by the following equations (Ovsiannikov [1960]):

$$(h\xi)_x + (h\eta)_y = 0, \quad (k\xi)_x + (k\eta)_y = 0, \quad (3.22)$$

$$(\sigma + a\eta + b\xi)_x = (h - k)\eta, \quad (\sigma + a\eta + b\xi)_y = (k - h)\xi, \quad (3.23)$$

or (Ibragimov [1992])

$$d(h\Xi) = 0, \quad d(k\Xi) = 0, \quad d(\sigma + a\eta + b\xi) = (h - k)\Xi, \quad (3.24)$$

where

$$\Xi = \eta dx - \xi dy \quad (3.25)$$

is the *dual differential one-form* for Operator 3.21.

## 3.2. LIE GROUP TREATMENT OF RIEMANN'S METHOD

This section is a synthesis Riemann's method (Riemann [1860]) for integration of linear hyperbolic second order differential equations in two variables with the group classification of these equations due to Lie [1881]. The comprehensive analysis of the well-known methods of construction of the Riemann function (alias Riemann–Green function) is presented in Copson

[1957] for the special types of equations. As a whole, six different methods are described in Copson's paper. The seventh method based on Lie group analysis is offered in Ibragimov [1991], [1992].

### 3.2.1. INVARIANCE PRINCIPLE IN BOUNDARY (INITIAL) VALUE PROBLEMS

Many problems of mathematical physics can be solved by utilizing the following semi-empirical principle.

#### INVARIANCE PRINCIPLE

*If a boundary value problem is invariant with respect to a group  $G$  then a solution of the problem should be looked for in the class of invariant functions under  $G$ .*

*Invariance of a boundary problem for a given differential equation involves the invariance (1) of the differential equation under consideration, (2) of the boundary (or initial) manifold, and (3) of the data to the problem under the action of  $G$  on the boundary manifold.*

If boundary conditions of the original problem are not invariant, the invariance principle is useful in a combination with other methods. For example, Riemann's method reduces the Cauchy problem with an arbitrary (and hence noninvariant) data to a special Goursat problem solvable by the invariance principle.

### 3.2.2. THE RIEMANN FUNCTION

Riemann's method reduces the integration problem for the equation

$$L[u] \equiv u_{xy} + a(x, y)u_x + b(x, y)u_y + c(x, y)u = f(x, y) \quad (3.26)$$

to the construction of the auxiliary function  $v$  defined by the adjoint equation

$$L^*[v] \equiv v_{xy} - (av)_x - (bv)_y + cv = 0 \quad (3.27)$$

together with the following conditions on the characteristics:

$$v|_{x=x_0} = \exp \int_{y_0}^y a(x_0, \eta) d\eta, \quad v|_{y=y_0} = \exp \int_{x_0}^x b(\xi, y_0) d\xi. \quad (3.28)$$

The boundary-value problem 3.27-3.28 is known as the characteristic Cauchy problem, or the Goursat problem. There exists the unique classical solution of this problem. The solution  $v$  to the Problem 3.27-3.28 is known as the Riemann function. If the function  $v$  is found, the solution of the Cauchy problem

$$L(u) = f, \quad u|_\gamma = u_0(x), \quad u_y|_\gamma = u_1(x) \quad (3.29)$$

with the data on the arbitrary noncharacteristic curve  $\gamma$  is given by the integral formula

$$\begin{aligned} u(x_0, y_0) = & \frac{1}{2}[u(A)v(A) + u(B)v(B)] \\ & + \int_{AB} \left\{ \left[ \frac{1}{2}vu_x + (bv - \frac{1}{2}v_x)u \right] dx - \left[ \frac{1}{2}vu_y + (av - \frac{1}{2}v_y)u \right] dy \right\} \\ & + \iint v f dx dy, \end{aligned}$$

where A and B are the intersections of  $\gamma$  with the characteristics  $y = y_0$  and  $x = x_0$ , respectively, and the twofold integral is taken over the region bounded by the characteristics  $x = x_0$ ,  $y = y_0$  and the curve  $\gamma$ .

### 3.2.3. EXAMPLES FOR SOLUTION OF THE GOURSAT PROBLEM

**Example 1.** For the wave equation  $u_{xy} = 0$  the corresponding Goursat problem has an especially simple form

$$v_{xy} = 0, \quad v|_{x=x_0} = 1, \quad v|_{y=y_0} = 1.$$

Evidently the solution of the problem is  $v = 1$ .

**Example 2.** The telegraph equation

$$u_{xy} + u = 0, \tag{3.30}$$

is the simplest equation (following equation  $u_{xy} = 0$ ) which allows the application of the Riemann method. The Goursat problem (3.27)-(3.28), in this case, is as follows:

$$v_{xy} + v = 0, \quad v|_{x=x_0} = 1, \quad v|_{y=y_0} = 1. \tag{3.31}$$

The standard textbooks suggest to solve the problem by letting

$$v = V(z), \quad \text{where } z = (x - x_0)(y - y_0). \tag{3.32}$$

Then Problem 3.31 reduces to the following problem for second-order ordinary differential equation:

$$zV'' + V' + V = 0, \quad V(0) = 1.$$

This is the Bessel equation. The substitution  $\mu = \sqrt{4z}$  transforms it to the standard form  $\mu V'' + V' + \mu V = 0$ .

Thus, the Riemann function for the telegraph equation is given by the Bessel function

$$v(x, y; x_0, y_0) = J_0\left(\sqrt{4(x - x_0)(y - y_0)}\right). \tag{3.33}$$

**Example 3.** Riemann applied his method to the equation

$$v_{xy} + \frac{l}{(x+y)^2}v = 0, \quad l = \text{const.} \quad (3.34)$$

Here, Equations (3.28) have the form

$$v|_{x=x_0} = 1, \quad v|_{y=y_0} = 1. \quad (3.34')$$

By letting

$$v = V(z), \quad \text{where } z = \frac{(x-x_0)(y-y_0)}{(x_0+y_0)(x+y)}, \quad (3.35)$$

Riemann [1860] reduces the characteristic Cauchy problem to an ordinary differential equation.

Guessing an appropriate form of the solution to the Goursat problem, in the last two examples, we have reduced the problem of construction of the Riemann function to the solution of an ordinary differential equation. It appears that this reduction follows from the existence of a symmetry group for the Goursat problem. It's just the invariance principle, and not the successful guess, that gives the correct form of a solution. Let's consider the equations presented in Examples 2 and 3 from this point of view.

### 3.2.4. ILLUSTRATIONS TO THE INVARIANCE PRINCIPLE

**Example 1.** Equation 3.30 admits the three-parameter group with the generators

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}.$$

Let us find the linear combination of these operators

$$X = \alpha X_1 + \beta X_2 + \gamma X_3,$$

admitted by the Goursat problem.

Characteristics  $x = x_0$  and  $y = y_0$  are invariant if

$$[X(x-x_0)]|_{x=x_0} \equiv \alpha + \gamma x_0 = 0, \quad [X(y-y_0)]|_{y=y_0} \equiv \beta - \gamma y_0 = 0.$$

It follows that  $\gamma \neq 0$  (otherwise  $\alpha = \beta = 0$ ). Hence, we can set  $\gamma = 1$  and get  $\alpha = -x_0$ ,  $\beta = y_0$ . The resulting operator

$$X = (x-x_0) \frac{\partial}{\partial x} - (y-y_0) \frac{\partial}{\partial y}, \quad (3.36)$$

is obviously admitted by Equations 3.31. Thus, we can employ the invariance principle and look for the solution of the problem in the class of invariant functions under the transformations generated by Operator 3.36. This group has two independent invariants, viz.  $v$  and  $z = (x - x_0)(y - y_0)$ . Hence, the invariant solution has the form (3.32).

**Example 2.** Equation 3.34 also admits three operators (in addition to dilations and the infinite dimensional group of linear equation):

$$X_1 = \frac{\partial}{\partial x} - \frac{\partial}{\partial y}, \quad X_2 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, \quad X_3 = x^2 \frac{\partial}{\partial x} - y^2 \frac{\partial}{\partial y}.$$

As in the previous case, they can be grouped in a linear combination, namely:

$$X = (x - x_0)(x + y_0) \frac{\partial}{\partial x} - (y - y_0)(y + x_0) \frac{\partial}{\partial y}, \quad (3.37)$$

that lives invariant characteristics and the conditions on them. Thus, we should look for the solution of Problem 3.34-3.34' in the class of invariant functions. The invariants for Operator 3.37 are  $v$  and

$$\mu = \frac{(x - x_0)(y - y_0)}{(x + y_0)(y + x_0)} \quad (3.38)$$

and the invariant solution has the form  $v = v(\mu)$ . This solution coincides with that found by Riemann; the variable  $z$  that he used is expressed in terms of the invariant  $\mu$  by the functional relationship  $z = \mu/(1 - \mu)$  and therefore is an invariant as well.

### 3.2.5. LAPLACE INVARIANTS AND THE GROUP CLASSIFICATION

We proceed from the group classification due to Lie [1881] for homogeneous equations (3.26). For our purposes, it is convenient to use the invariant formulation suggested by Ovsiannikov [1960] (see also [H1], Section 9.1.2.1).

**Theorem 3.1.** The equation

$$L[u] \equiv u_{xy} + a(x, y)u_x + b(x, y)u_y + c(x, y)u = 0 \quad (3.39)$$

admits a four-dimensional Lie algebra  $L_4$  if and only if the invariants (Ovsiannikov's invariants)

$$p = k/h, \quad q = (\ln|h|)_{xy}/h \quad (3.40)$$

(if  $h = 0$ , one should replace  $h$  and  $k$ ) are constants, and  $L_r$ ,  $r \leq 2$ , if  $p$  or  $q$  (or both of them) is not constant. Here  $h$  and  $k$  are Laplace invariants of Equation 3.39:

$$h = a_x + ab - c, \quad k = b_y + ab - c. \quad (3.41)$$

In the case of constant  $p$  and  $q$  Equation 3.39 is reduced either to the form (if  $q = 0$ )

$$u_{xy} + xu_x + pyu_y + pxyu = 0 \quad (3.42)$$

and admits the operator

$$X = (\alpha_1 x + \alpha_2) \frac{\partial}{\partial x} + (-\alpha_1 y + \alpha_3) \frac{\partial}{\partial y} - (\alpha_2 y + \alpha_3 px + \alpha_4) u \frac{\partial}{\partial u}, \quad (3.43)$$

or to the form (if  $q \neq 0$ )

$$u_{xy} - \frac{2}{q(x+y)} u_x - \frac{2p}{q(x+y)} u_y + \frac{4p}{q^2(x+y)^2} u = 0 \quad (3.44)$$

and admits the operator

$$X = (\alpha_1 x^2 + \alpha_2 x + \alpha_3) \frac{\partial}{\partial x} + (-\alpha_1 y^2 + \alpha_2 x - \alpha_3) \frac{\partial}{\partial y} - \left[ \alpha_1 \frac{2}{q} (px - y) + \alpha_4 \right] u \frac{\partial}{\partial u}. \quad (3.45)$$

**Corollary.** The adjoint equation 3.27 admits a four-dimensional algebra if and only if this is true for Equation 3.39.

In fact, mutually adjoint equations,  $L[u] = 0$  and  $L^*[v] = 0$ , have the Laplace invariants  $h, k$  and  $h_* = k$ ,  $k_* = h$ . Hence, the corresponding invariants 3.40:  $p, q$  and  $p_*, q_*$  are constant or not simultaneously.

**Remark.** The general form of the equivalence transformations of Equations 3.39 is as follows:

$$\bar{x} = f(x), \quad \bar{y} = g(y), \quad \bar{u} = \lambda(x, y)u. \quad (3.46)$$

Laplace invariants 3.41 are in fact invariant only relative to linear transformations of the dependent variable  $u$  that do not change coordinates  $x$  and  $y$ . On the contrary, the quantities  $p$  and  $q$  defined by Formulae 3.40 are

invariant under the general equivalence transformations 3.46 and therefore naturally arise in Theorem 3.1.

### 3.2.6. THE MAIN RESULT

**Theorem 3.2 (Ibragimov [1992]).** Let Equation 3.26

$$L[u] \equiv u_{xy} + au_x + bu_y + cu = f$$

have constant Ovsiannikov's invariants 3.40:

$$p = k/h, \quad q = (\ln|h|)_{xy}/h$$

where  $h = a_x + ab - c$ ,  $k = b_y + ab - c$ . Then the Goursat problem 3.27-3.28:

$$L^*[v] = 0, \quad v|_{x=x_0} = \exp \int_{y_0}^y a(x_0, \eta) d\eta, \quad v|_{y=y_0} = \exp \int_{x_0}^x b(\xi, y_0) d\xi,$$

admits a one-parameter group and the Riemann function is obtained from the ordinary second-order differential equation.

**Proof.** It follows from Theorem 3.1 and its corollary that it is sufficient to consider Equation 3.26 such that its adjoint equation 3.27 has the form (3.42) or (3.44) depending on whether the invariant  $q_*$  is equal to zero or not.

Let  $q_* = 0$  and let the adjoint equation have the form (3.42):

$$L^*[v] \equiv v_{xy} + xv_x + p_*yv_y + p_*xyv = 0, \quad p_* = \text{const.} \quad (3.42^*)$$

Then conditions 3.28 are

$$v|_{x=x_0} = e^{x_0(y_0-y)}, \quad v|_{y=y_0} = e^{p_*y_0(x_0-x)}, \quad (3.42')$$

and the Goursat problem 3.42\*-3.42' admits the operator

$$X = (x - x_0) \frac{\partial}{\partial x} - (y - y_0) \frac{\partial}{\partial y} + [x_0(y - y_0) - p_*y_0(x - x_0)]v \frac{\partial}{\partial v}.$$

This operator has two invariants, viz.

$$V = ve^{[x_0(y-y_0)+p_*y_0(x-x_0)]} \quad \text{and} \quad \mu = (x - x_0)(y - y_0).$$

In accordance with the invariance principle we seek the solution of the Goursat problem in the invariant form

$$v = e^{x_0(y_0-y)+p_*y_0(x_0-x)}V(\mu), \quad \mu = (x - x_0)(y - y_0). \quad (3.47)$$

Substitution in (3.42\*) yields:

$$\mu V'' + (1 + (p_* + 1)\mu)V' + p_* \mu V = 0 \quad (3.48)$$

and the characteristic data 3.42' becomes  $V(0) = 1$ . Hence, the construction of the Riemann function is reduced to the solution of the Equation 3.48 with the initial condition  $V(0) = 1$ .

Let  $q_* \neq 0$  and let the adjoint equation have the form (3.44):

$$L_*[v] \equiv v_{xy} - \frac{2}{q_*(x+y)}v_x - \frac{2p}{q_*(x+y)}v_y + \frac{4p}{q_*^2(x+y)^2}v = 0, \\ p_*, q_* = \text{const.} \quad (3.44^*)$$

In this case Conditions 3.28 assume the form

$$v|_{x=x_0} = \left( \frac{x_0 + y}{x_0 + y_0} \right)^{2/q_*}, \quad v|_{y=y_0} = \left( \frac{x + y_0}{x_0 + y_0} \right)^{2p_*/q_*}, \quad (3.44')$$

and the Goursat problem 3.44\*-3.44' admits the operator

$$X = (x - x_0)(x + y_0)\frac{\partial}{\partial x} - (y - y_0)(x_0 + y)\frac{\partial}{\partial y} \\ + \frac{2}{q_*}[p_*(x - x_0) - (y - y_0)]v\frac{\partial}{\partial v}.$$

The invariants are

$$V = (x + y_0)^{-2p_*/q_*}(x_0 + y)^{-2/q_*}v \quad \text{and} \quad \mu = \frac{(x - x_0)(y - y_0)}{(x + y_0)(y + x_0)}.$$

Hence, the invariant solution is of the form

$$v = \left( \frac{x + y_0}{x_0 + y_0} \right)^{2p_*/q_*} \left( \frac{x_0 + y}{x_0 + y_0} \right)^{2/q_*} V(\mu), \quad \mu = \frac{(x - x_0)(y - y_0)}{(x + y_0)(y + x_0)}. \quad (3.49)$$

The Goursat problem 3.44\*-3.44' is then reduced to the solution of the ordinary differential equation

$$\mu(1 - \mu)^2 V'' + (1 - \mu) \left[ 1 + \left( \frac{2(p_* + 1)}{q_*} - 1 \right) \mu \right] V' + \frac{4p_*}{q_*^2} \mu V = 0 \quad (3.50)$$

with the condition  $V(0) = 1$ .

For the practical utilization of Theorem 3.2, there is no need to reduce the adjoint equation to the corresponding standard form (3.42\*) or (3.44\*).

### 3.2.7. EXAMPLE TO THE METHOD

Consider the equation

$$u_{xy} - \frac{l}{x+y}(u_x + u_y) = f(x, y), \quad l \neq 0, l \neq -1. \quad (3.51)$$

Here,  $h = k = l(l+1)(x+y)^{-2}$  and Formulae 3.40 give  $p = 1$ ,  $q = 2/l(l+1)$ . Hence, Theorem 3.2 can be applied.

For the adjoint equation

$$v_{xy} + \frac{l}{x+y}(v_x + v_y) - \frac{2l}{(x+y)^2}v = 0, \quad (3.52)$$

Equations 3.28 become

$$v|_{x=x_0} = \left( \frac{x_0 + y_0}{x_0 + y} \right)^l, \quad v|_{y=y_0} = \left( \frac{x_0 + y_0}{x + y_0} \right)^l. \quad (3.53)$$

Let us calculate the symmetry algebra for the Equation 3.52. Laplace invariants for Equation 3.52 are equal to each other and coincide with the invariants  $h = k$  for Equation 3.51 (see corollary of Theorem 3.1). Thus, the determining equations (3.22) reduce to the following equation:

$$\left( \frac{\eta}{(x+y)^2} \right)_y + \left( \frac{\xi}{(x+y)^2} \right)_x = 0,$$

or

$$(x+y)(\xi'(x) + \eta'(y)) = 2(\xi(x) + \eta(y)). \quad (3.54)$$

It follows:

$$\xi = C_1 x^2 + C_2 x + C_3, \quad \eta = -C_1 y^2 + C_2 y - C_3.$$

Then the third equation 3.24 has the form

$$d \left( \sigma + \frac{l}{x+y}(\xi + \eta) \right) = 0,$$

and yields

$$\sigma = C_4 + C_1 l(y - x).$$

Thus, the symmetry generator is given by

$$X = (C_1x^2 + C_2x + C_3)\frac{\partial}{\partial x} - (C_1x^2 - C_2y + C_3)\frac{\partial}{\partial y} + [C_4 + C_1l(y-x)]v\frac{\partial}{\partial v} \quad (3.55)$$

and contains four arbitrary constants  $C_1, C_2, C_3, C_4$  in accordance with Theorem 3.1.

Now we choose constants in Operator 3.55 from the invariance conditions for the data (3.53). The invariance of the characteristics  $x = x_0$  and  $y = y_0$  implies

$$[X(x - x_0)]|_{x=x_0} \equiv C_1x_0^2 + C_2x_0 + C_3 = 0,$$

$$[X(y - y_0)]|_{y=y_0} \equiv -C_1y_0^2 + C_2y_0 - C_3 = 0.$$

Hence,  $C_2 = (y_0 - x_0)C_1$ ,  $C_3 = -x_0y_0C_1$ . We choose  $C_1 = 1$ , and obtain

$$X = (x - x_0)(x + y_0)\frac{\partial}{\partial x} - (y - y_0)(x_0 + y)\frac{\partial}{\partial y} + [C_4 + l(y - x)]v\frac{\partial}{\partial v}. \quad (3.56)$$

The invariance condition for the first Equation 3.53 yields  $C_4 = l(x_0 - y_0)$ . Then the second Equation 3.53 is also invariant. Hence, Goursat problem 3.52-3.53 admits the one-parameter group with the symbol

$$X = (x - x_0)(x + y_0)\frac{\partial}{\partial x} - (y - y_0)(x_0 + y)\frac{\partial}{\partial y} + l[(y - y_0) - (x - x_0)]v\frac{\partial}{\partial v}. \quad (3.57)$$

This group has two functionally independent invariants:

$$z = \frac{(x - x_0)(y - y_0)}{(x_0 + y_0)(x + y)} \quad \text{and} \quad V = (x_0 + y_0)^{-2l}(x + y_0)^l(y + x_0)^l v. \quad (3.58)$$

Hence, the invariant solution can be written

$$v = \frac{(x_0 + y_0)^{2l}}{(x + y_0)^l (y + x_0)^l} V(z). \quad (3.59)$$

Substituting in Equation 3.52 yields:

$$z(1 + z)V'' + (1 + 2(1 - l)z)V' - l\left(2 + \frac{l}{1 + z}\right)V = 0. \quad (3.60)$$

Two conditions on characteristics reduce to  $V(0) = 1$ .

# Huygens' Principle: Conformal Invariance, Darboux Transformation, and Coxeter Groups

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This chapter provides an elucidating glimpse at group theoretic aspects of Huygens' principle for linear hyperbolic second order partial differential equations.

A general theory, a historical survey, and an extensive bibliography concerned with Huygens' principle are presented in Hadamard [1923], McLenaghan [1982], Ibragimov [1983], and Günther [1988].

A detailed group analysis of Huygens' operators and an account of results obtained prior to 1983 are to be found in Ibragimov [1970], [1983]. Recently a number of results has been obtained (Ibragimov and Oganessian [1991]; Berest, Ibragimov, and Oganessian [1993]; Berest [1993a]; Berest and Veselov [1993a], [1993b]) that reveal an interdependence of Hadamard's treatment of Huygens' principle and contemporary theoretical concepts recognized as being valid in soliton mathematics.

Along with classical results, we outline here the new developments.

## 4.1. HUYGENS' DIFFERENTIAL EQUATIONS: HADAMARD'S PROBLEM

Given the hyperbolic differential equation

$$F[u] = g^{ij}(x)u_{ij} + b^i(x)u_i + c(x)u = 0 \quad (4.1)$$

in  $n$  independent variables  $x = (x^1, \dots, x^n)$ , Cauchy's problem is a problem of determining a solution which assumes given values of  $u$  and its normal derivative on a space-like  $(n - 1)$ -dimensional surface  $S$ . These given values are called the Cauchy data. According to our local approach, the problem is considered in a neighborhood of a generic point  $x_0$ , where the coefficients of Equation 4.1 are assumed to be  $\mathbb{C}^\infty$ -smooth, and  $\det g^{ij} \neq 0$ .

In general, the solution of the initial value problem at a point  $x_0$  depends on the Cauchy data in the interior of the intersection  $S \cap C^-(x_0)$  of the initial surface  $S$  and the retrograde characteristic conoid  $C^-(x_0)$  with a vertex  $x_0$ .

If the solution depends only on the data in an arbitrary small neighborhood of  $S \cap C^-(x_0)$  for every Cauchy's problem and for every  $x_0$  we say (after Hadamard [1923]) that Equation 4.1 satisfies Huygens' principle or is a Huygens' differential equation. This is equivalent to the statement that the fundamental solution for Equation 4.1 has the support only on the surface of the characteristic conoid.

The physical significance of Huygens' principle is that waves governed by Huygens' operators propagate without diffusion.

Familiar examples of Huygens' operators are provided by the ordinary wave equations with an even number  $n \geq 4$  of independent variables.

The problem of determining all the Huygens' differential equations 4.1 is known in the literature as Hadamard's problem.

## 4.2. ELEMENTARY SOLUTION: HADAMARD'S CRITERION AND PAINLEVÉ PROPERTY

The necessary condition for the validity of Huygens' principle is that  $n \geq 4$  be even (Hadamard [1923]).

Accordingly, we set in Equation 4.1  $n = 2(p + 1)$ ,  $p \geq 1$ . Then an elementary solution for the adjoint equation:

$$F^*[u] = (g^{ij}v)_{ij} - (b^i v)_i + cv = 0 \quad (4.1')$$

is written in the form (Hadamard [1923], for generalizations see, e.g., Babitch [1991]):

$$v(x, x_0) = V(x, x_0)\Gamma^{-p} - W(x, x_0)\log \Gamma + w, \quad (4.2)$$

where  $V(x, x_0) = \sum_{\nu=0}^{p-1} U_\nu(x, x_0)\Gamma^\nu$ ,  $W(x, x_0) = \sum_{\nu=p}^\infty U_\nu(x, x_0)\Gamma^{\nu-p}$  and  $w(x)$  is a regular function. This is a singular solution with singularities on the characteristic conoid given by

$$\Gamma(x, x_0) = 0. \quad (4.3)$$

Here,  $\Gamma(x, x_0)$  is the square of the geodesic distance from  $x$  to  $x_0$  in the Riemannian space  $V_n$  associated with Equation 4.1 (see Chapter 3). It solves, as a function of  $x$ , the following characteristic equation:

$$g^{ij}(x)\Gamma_i\Gamma_j = 4\Gamma. \quad (4.4)$$

The coefficients  $U_\nu(x, x_0)$  are single-valued regular functions known as Hadamard coefficients.

Hadamard's criterion asserts (Hadamard [1923], Section 129) that the necessary and sufficient condition of the validity of Huygens' principle for the Equation 4.1 is that the elementary solution 4.2 does not contain any logarithmic term. This condition can be equivalently written as follows:

$$U_\nu|_{\Gamma=0} = 0, \quad \nu \geq p. \quad (4.5)$$

Thus, Equation 4.1 satisfies Huygens' principle if and only if its adjoint equation 4.1' obeys a conditional Painlevé property according to the terminology employed in Weiss [1984] (see also Weiss, Tabor, and Carnevale [1983]).

### 4.3. CONFORMAL TREATMENT OF HUYGENS' EQUATIONS IN $V_4$

"We have enunciated the necessary and sufficient condition, but we do not know how equations satisfying it can be found... This, and many other questions concerning the residual integral, would require further researchers" (Hadamard [1923], Section 149).

An explicit determination of all Huygens' equations for arbitrary even  $n$  is still an open problem. However, the solution of the problem has considerably advanced in the case  $n = 4$ , which is of the special interest because of its physical significance.

#### 4.3.1. WAVE EQUATION IN SPACE-TIMES WITH NONTRIVIAL CONFORMAL GROUP

Mathisson [1939] and Asgeirsson [1956] proved independently that the classical wave equation is the only (up to trivial transformations) Huygens' equation in Minkowski space. The following theorem (Ibragimov [1970]) extends Mathisson-Asgeirsson's result and solves Hadamard's problem on all the spaces  $V_4$  with nontrivial conformal group.

Let the Riemannian space  $V_4$  associated with Equation 4.1 be a space-time with nontrivial conformal group. Then Huygens' principle holds if and only if Equation 4.1 is conformally invariant, i.e., equivalent to the wave equation in  $V_4$ :

$$\square u = g^{ij}u_{,ij} + \frac{1}{6}Ru = 0. \quad (4.6)$$

For the proof, see Ibragimov [1970] and [1983].

### 4.3.2. REALIZATION ON THE PLANE WAVE SPACE-TIME

Any space-time with nontrivial conformal group can be reduced, by a proper conformal mapping and a choice of coordinates, to the plane wave space-time. In this realization, Equation 4.6 is equivalent to the following one

$$u_{tt} - u_{xx} - f(x-t)u_{yy} - 2\varphi(x-t)u_{yz} - u_{zz} = 0, \quad (4.7)$$

where  $f$  and  $\varphi$  depend on a single variable  $x-t$  and satisfy the hyperbolicity condition  $f - \varphi^2 > 0$ .

The validity of Huygens' principle for Equation 4.7 was revealed independently by Günther [1965] and Ibragimov and Mamontov [1970] (see also Ibragimov [1970]).

A detailed discussion of the solution formula to Cauchy's problem for Equation 4.7 and to its extension to arbitrary  $n$  is given in Ibragimov and Mamontov [1977] and Ibragimov [1983].

### 4.3.3. THE CASE OF TRIVIAL CONFORMAL GROUP

McLenaghan [1969], [1974] and Wünsch [1978], [1979], using necessary conditions for the validity of Huygens' principle (derived from Hadamard's criterion), investigated peculiar attributes of spaces  $V_4$  associated with Huygens' equations. They demonstrate that the validity of Huygens' principle for Equation 4.1 on a space-time satisfying certain supplementary conditions (conformally empty, Einstein space-times, symmetric, recurrent, etc.) implies that  $V_4$  is conformally equivalent to flat or a plane wave space-times, these being the only known space-times where Huygens' equations do exist. An outline of these results is to be found in McLenaghan [1982].

### 4.3.4. CONJECTURE ON HUYGENS' PRINCIPLE IN $V_4$

At present, the following question becomes a fundamental one in solving Hadamard's problem:

Are there spaces  $V_4$  with trivial conformal group such that Huygens' principle is satisfied by any of Equations 4.1 in these spaces?

We expect a negative answer to this question when dealing with differential equations in the real domain.

An interesting candidate to be a counterexample to this conjecture in the complex domain is presented in McLenaghan [1982], page 225.

#### 4.4. DARBOUX-LAGNESE-STELLMACHER TRANSFORMATION

In higher dimensions, the situation is more mysterious because of the failure of the relation of Huygens' principle to the conformal invariance.

A hierarchy of Huygens' equations that are not conformally invariant, is derived by the following construction due to Lagnese and Stellmacher [1967].

In  $(n + 1)$ -dimensional Minkowski space, we consider operators of the form:

$$F_{n+1} = \square_{n+1} + c \equiv \frac{\partial^2}{\partial t^2} - \sum_{i=1}^n \frac{\partial^2}{(\partial x^i)^2} + c(x, t), \quad (4.8)$$

where

$$c(x, t) = \alpha_0(t) - \alpha_1(x^1) - \cdots - \alpha_n(x^n). \quad (4.9)$$

Then we define

$$\tilde{c}(x, t) = \tilde{\alpha}_0(t) - \tilde{\alpha}_1(x^1) - \cdots - \tilde{\alpha}_n(x^n) \quad (4.9')$$

by the following Darboux transformations:

$$\tilde{\alpha}_k(x^k) = -\alpha_k(x^k) - 2\beta_k^2(x^k), \quad (4.10)$$

where  $k = 0, 1, \dots, n$ ;  $x^0 = t$ , and the functions  $\beta_k$  are determined via  $n + 1$  Riccati equations:

$$\frac{d}{dx^k} \beta_k + \beta_k^2 + \alpha_k(x^k) = 0, \quad k = 0, 1, \dots, n. \quad (4.11)$$

Now we proceed from any Huygens' operator of the form (4.8) (e.g., take  $c \equiv 0$ , and  $n \geq 3$  odd) and perform Darboux-Lagnese-Stellmacher transformations 4.9'-4.11 successively, one step being the transformation of any one of the coefficients  $\alpha_k$  only. Then, according to Lagnese-Stellmacher's theorem, after  $s$  steps we obtain the Huygens' operator in  $(n + 1 + 2s)$ -dimensional Minkowski space,

$$F_{n+2s+1} = \square_{n+1+2s} + \tilde{c}. \quad (4.12)$$

For example,  $F_{3+1} = \square_{3+1}$  becomes, after the one-step transformation of  $\alpha_0(t)$ , as follows:

$$F_{5+1} = \square_{5+1} - \frac{2}{t^2}, \quad (4.13)$$

and after the second step that we choose to be the transformation of  $\alpha_1(x^1)$ , we obtain the following Huygens' operator:

$$F_{7+1} = \square_{7+1} - \frac{2}{t^2} + \frac{2}{(x^1)^2}. \quad (4.13')$$

Although the Lagnese–Stellmacher huygensian operators fail to be conformally invariant, the whole hierarchy of such operators admits the nontrivial Lie–Bäcklund algebra of equivalence transformations preserving Huygens' principle (Berest [1993]). This algebra is shown to be intimately related to the higher symmetries and rational solutions of the KdV-equation in the theory of solitons.

## 4.5. NEW HUYGENS' OPERATORS IN FLAT SPACES RELATED TO COXETER GROUPS

Quite recently a new class of Huygens' operators in multi-dimensional Minkowski spaces has been discovered (Berest and Veselov [1993a], [1993b]). They happen to generalize Stellmacher's examples (see, e.g., Equations 4.12 and 4.13) to the case of arbitrary root systems.

### 4.5.1. ROOT SYSTEMS AND COXETER GROUPS

Let  $E^n$  be an Euclidean space of finite dimension  $n$ . With any nonzero vector  $\alpha \in E^n$  the following linear operator is associated:

$$s_\alpha: E^n \rightarrow E^n, \quad x \mapsto s_\alpha(x) = x - \frac{2(\alpha, x)}{(\alpha, \alpha)} \alpha, \quad (4.14)$$

where  $(\alpha, x)$  is a scalar product of the vectors  $\alpha$  and  $x$  in the space  $E^n$ . The operator  $s_\alpha$  provides a reflection of the space  $E^n$  with respect to the hyperplane  $(\alpha, x) = 0$  normal to the vector  $\alpha$ .

Consider a finite set of unit vectors  $\mathfrak{R} = \{\alpha\} \in E^n$ , satisfying the following conditions: (i) the set  $\mathfrak{R}$  generates the space  $E^n$ , i.e.,  $E^n = \text{span}\langle \mathfrak{R} \rangle$ , and (ii) for every  $\alpha \in \mathfrak{R}$  reflection  $s_\alpha$  conserves  $\mathfrak{R}$ :  $s_\alpha(\mathfrak{R}) = \mathfrak{R}$ . Such a system of vectors  $\mathfrak{R}$  is called the root system in the space  $E^n$  and the related reflections  $s_\alpha$ ,  $\alpha \in \mathfrak{R}$ , generate a finite group which is referred to as the Coxeter group of the rank  $n$ . A complete classification of Coxeter groups is to be found in Bourbaki [1968].

### 4.5.2. NEW HUYGENS' OPERATORS RELATED TO ROOT SYSTEMS

Let  $\mathfrak{R}$  be a root system of the rank  $n$  and  $W$  be a Coxeter group associated with  $\mathfrak{R}$ . In  $(1 + N)$ -dimensional Minkowski space  $M^{N+1}$  one

considers the following hyperbolic differential operator:

$$\left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y_1^2} - \cdots - \frac{\partial^2}{\partial y_m^2} - \frac{\partial^2}{\partial x_1^2} - \cdots - \frac{\partial^2}{\partial x_n^2} + c(x) \right) u = 0 \quad (4.15)$$

with the potential

$$c(x) = \sum_{\alpha \in \mathfrak{R}} g_{\alpha} \frac{(\alpha, \alpha)}{(x, \alpha)^2}, \quad g_{\alpha} = \frac{m_{\alpha}(m_{\alpha} + 1)}{2}. \quad (4.16)$$

Here,  $(x_1, \dots, x_n, y_1, \dots, y_m)$  is a set of spatial variables, so that  $N = n + m$ ;  $m_{\alpha}$  is a  $W$ -invariant integer-valued function on  $\mathfrak{R}$ :

$$m: \mathfrak{R} \rightarrow \mathbb{Z}_+.$$

The following theorem is valid:

For any root system  $\mathfrak{R}$  the hyperbolic operators 4.15 with the potential 4.16 satisfy Huygens' principle if  $N$  is odd and

$$N \geq 3 + \sum_{\alpha \in \mathfrak{R}} m_{\alpha}. \quad (4.17)$$

For example, the following two operators:

$$F_{A_2} = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{(\partial y^1)^2} - \cdots - \frac{\partial^2}{(\partial y^8)^2} - \frac{2}{t^2} - \frac{16(t^2 - 3x^2)}{(t^2 + 3x^2)^2},$$

$$F_{B_2} = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{(\partial y^1)^2} - \cdots - \frac{\partial^2}{(\partial y^{10})^2} - \frac{2}{t^2} + \frac{2}{x^2} - \frac{8(t^2 - x^2)}{(t^2 + x^2)^2}$$

respectively related to the Coxeter group of  $A_2$  and  $B_2$  types satisfy Huygens' principle (their fundamental solutions are presented in Chapter 7 of the present book).

The elliptic operators of the form

$$L = -\Delta + c(x), \quad \Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}, \quad (4.18)$$

where the potential  $c(x)$  is given by the Formula 4.16 with the parameter  $g_{\alpha}$  assumed not necessary to be integer, have been introduced by Olshanetsky and Perelomov [1977] as a natural generalization of Calogero operators (Calogero [1971]) within the framework of the theory of integrable systems.

The case of the special values of the parameter  $g_{\alpha}$  given in Formula 4.16 have been investigated by Chalykh and Veselov [1990], [1992], who showed that such operators possess additional nonpoint symmetries and eigenfunctions with remarkable algebraic properties.

# Applications to Celestial Mechanics and Astrophysics

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## 5.1. NEWTON-COTES POTENTIAL

Consider a motion of a particle with mass  $m$  in a potential central field. We write the Lagrangian in the form

$$L = \frac{m}{2} \mathbf{v}^2 + U(r), \quad \left( \mathbf{v} = \dot{\mathbf{x}} \equiv \frac{d\mathbf{x}}{dt} \right). \quad (5.1)$$

The equation of motion is given by

$$m\ddot{\mathbf{x}} = -\frac{U'(r)}{r} \mathbf{x}. \quad (5.2)$$

Having in mind the importance of scaling transformations in mechanics, we would like to clarify the form of potentials  $U(r)$  for which the dilation group with the generator

$$T = kt \frac{\partial}{\partial t} + x^i \frac{\partial}{\partial x^i}, \quad k = \text{const}, \quad (5.3)$$

satisfies the Noether theorem on the conservation laws (i.e., when Operator 5.3 is a Noether symmetry).

It's convenient to represent generator 5.3 in the form of a canonical Lie-Bäcklund operator (Ibragimov [1983])

$$X = (x^i - ktv^i) \frac{\partial}{\partial x^i} \quad (5.4)$$

and write down an infinitesimal transformation of coordinates  $\bar{\mathbf{x}} \approx \mathbf{x} + \delta \mathbf{x}$ , where

$$\delta \mathbf{x} = (\mathbf{x} - kt\mathbf{v})a. \quad (5.5)$$

Differentiation of Equation 5.5 with respect to  $t$  yields the velocity increment

$$\delta \mathbf{v} = ((1 - k)\mathbf{v} - kt\dot{\mathbf{v}})a. \quad (5.6)$$

The principal part of the increment for the Lagrangian 5.1, under small variations  $\delta \mathbf{x}$  and  $\delta \mathbf{v}$  of the coordinates and the velocity, is

$$\delta L = m\mathbf{v} \cdot \delta \mathbf{v} + \frac{U'(r)}{r} \mathbf{x} \cdot \delta \mathbf{x}. \quad (5.7)$$

The Noether theorem can be formulated in our case as follows:

If the increment of the Lagrangian is a total derivative, viz.

$$\delta L = \frac{dF}{dt}, \quad (5.8)$$

then Equation 5.2 has the conserved quantity given by

$$J = m\mathbf{v} \cdot \delta \mathbf{v} - F. \quad (5.9)$$

**Theorem 5.1.** (Ibragimov [1993]). For Lagrangian 5.1 and infinitesimal transformations 5.5 and 5.6 of the dilation group Equation 5.8 is valid if and only if

$$U = -\frac{\alpha}{r^2}, \quad \alpha = \text{const.} \quad (5.10)$$

The corresponding equation of motion,

$$m\ddot{\mathbf{x}} = 2\alpha \frac{\mathbf{x}}{r^4}, \quad (5.11)$$

admits the dilation group with the generator

$$T = 2t \frac{\partial}{\partial t} + x^i \frac{\partial}{\partial x^i}. \quad (5.12)$$

Moreover, it also admits the group of projective transformations with the generator

$$Z = t^2 \frac{\partial}{\partial t} + tx^i \frac{\partial}{\partial x^i}. \quad (5.13)$$

Both of these operators satisfy Equation 5.8, and Formula 5.9 yields the corresponding conserved quantities:

$$J_1 = 2tE - m\mathbf{x} \cdot \mathbf{v}, \quad J_2 = 2t^2E - m\mathbf{x} \cdot (2t\mathbf{v} - \mathbf{x}), \quad (5.14)$$

where

$$E = \frac{m}{2} \mathbf{v}^2 + \frac{\alpha}{r^2}$$

is the energy.

**Proof.** In virtue of Equations 5.5, 5.6, and 5.2, Formula 5.7 is rewritten as

$$\delta L = \frac{d}{dt} ((1 - k)m\mathbf{x} \cdot \mathbf{v} - 2ktU)a + ka(rU' + 2U).$$

Therefore we have to find out when the expression  $rU' + 2U$  is a total derivative. This is possible if and only if the variational derivative of this expression vanishes:

$$\frac{\delta}{\delta x^i} (rU' + 2U) = \frac{\partial}{\partial x^i} (rU' + 2U) = (rU' + 2U)' \frac{x^i}{r} = 0.$$

Thus, for an unknown function  $U(r)$  we obtain a second order equation  $(rU' + 2U)' = 0$ . The solution of the latter, up to an additive constant, is given by Potential 5.10. The remaining statements of the theorem are checked by standard calculations (see also Ibragimov [1983], Section 25.2).

The central potential 5.10 possesses a number of specific properties (owing to its projective invariance) and thus arises in various problems of Newtonian mechanics. Newton in his *Principia* treated the motion of a body in this field. Then the motion of this kind was in greater details considered by Roger Cotes (1682–1716) in his “*Harmonia Mensurarum*.”

There also exists a mysterious relation between the Newton–Cotes potential 5.10 and the monatomic gas. This correlation is revealed via conservation laws 5.14 (see Ibragimov [1983], Section 25.2), that are also valid for Boltzman equations (Bobylov and Ibragimov [1989]).

## 5.2. GROUP THEORETIC NATURE OF THE KEPLER LAWS

It is known after Laplace [1798] that the first Kepler law (planets move on elliptic orbits with the Sun being in their focus) is the direct consequence of the conservation law of the vector<sup>1</sup>

$$\mathbf{A} = \mathbf{v} \times \mathbf{M} + \alpha \frac{\mathbf{x}}{r}, \quad \text{where } \mathbf{M} = m(\mathbf{x} \times \mathbf{v}), \quad (5.15)$$

for the equations

$$m\dot{\mathbf{v}} = \alpha \frac{\mathbf{x}}{r^3} \quad (5.16)$$

of motion in the central field with the potential

$$U = -\frac{\alpha}{r}, \quad \alpha = \text{const.} \quad (5.17)$$

Further, Equation 5.16 admits the infinitesimal Lie-Bäcklund transformation  $\delta \bar{\mathbf{x}} \approx \mathbf{x} + \delta \mathbf{x}$  with the vector-parameter  $\mathbf{a} = (a^1, a^2, a^3)$ , where

$$\delta \mathbf{x} = 2(\mathbf{a} \cdot \mathbf{x})\mathbf{v} - (\mathbf{a} \cdot \mathbf{v})\mathbf{x} - (\mathbf{x} \cdot \mathbf{v})\mathbf{a}, \quad (5.18)$$

and the Hermann-Bernoulli-Laplace vector 5.15 is obtained from the Noether theorem using Formula 5.9 (see Ibragimov [1983], Section 25.2).

This Lie-Bäcklund symmetry is a natural generalization of the rotational symmetry of the Kepler problem. In fact, one should only have to notice that the infinitesimal transformation of a group of rotations is defined by the increment  $\delta \mathbf{x} = \mathbf{x} \times \mathbf{a}$  and that Expression 5.18 can be presented in the form of a symmetric double vector product of vectors  $\mathbf{x}, \mathbf{v}, \mathbf{a}$ :

$$\delta \mathbf{x} = (\mathbf{x} \times \mathbf{v}) \times \mathbf{a} + \mathbf{x} \times (\mathbf{v} \times \mathbf{a}). \quad (5.19)$$

The invariance under the Lie-Bäcklund group with infinitesimal increment 5.19 is the group theoretic formulation of the first Kepler law.

**Theorem 5.2.** (Ibragimov [1993]). For a central field  $U(r)$ , infinitesimal transformation 5.19 satisfies Condition 5.8 for the applicability of the Noether theorem if and only if  $U(r)$  is Newton's potential 5.17.

**Proof.** We repeat arguments of the previous section. Here

$$\delta L = 2 \left[ \frac{d}{dt} ((\mathbf{x} \cdot \mathbf{a})U) - (\mathbf{v} \cdot \mathbf{a})(rU' + U) \right].$$

<sup>1</sup>According to Goldstein [1975], [1976] this vector has first appeared in 1710 as a constant of integration of the equations for orbit in publications by Jacob Hermann and Johann Bernoulli.

**TABLE 5.1**  
**Kepler Laws and Related Symmetries (*R* Is Semimajor Axis,  
and *T* Is the Orbital Period)**

Kepler Laws	Conservation Laws	Infinitesimal Symmetries
1. Orbit is an ellipse	Laplace vector $\mathbf{A} = \mathbf{v} \times \mathbf{M} + \alpha \frac{\mathbf{x}}{r}$	Lie-Bäcklund symmetry $\delta \mathbf{x} = \mathbf{x} \times (\mathbf{v} \times \mathbf{a}) + (\mathbf{x} \times \mathbf{v}) \times \mathbf{a}$
2. Conservation of area integral  $T \sim R^{3/2}$	$\mathbf{M} = m(\mathbf{x} \times \mathbf{v})$  —	Rotations $\delta \mathbf{x} = \mathbf{x} \times \mathbf{a}$  Dilations $\bar{t} = a^3 t, \bar{\mathbf{x}} = a^2 \mathbf{x},$ or $\delta \mathbf{x} = (2\mathbf{x} - 3t\mathbf{v})a,$ $t^2/r^3$ is invariant

Therefore Condition 5.8 is written as

$$\frac{\delta \Phi}{\delta \mathbf{x}} \equiv \frac{\partial \Phi}{\partial \mathbf{x}} - \frac{d}{dt} \frac{\partial \Phi}{\partial \mathbf{v}} = 0, \quad \text{where } \Phi = (\mathbf{v} \cdot \mathbf{a})(rU' + U).$$

Hence

$$\begin{aligned} \frac{\delta \Phi}{\delta \mathbf{x}} &= (rU' + U)'(\mathbf{v} \cdot \mathbf{a}) \frac{\mathbf{x}}{r} - \frac{d}{dt}(rU' + U)\mathbf{a} \\ &= \frac{1}{r}(rU' + U)' \mathbf{v} \times (\mathbf{x} \times \mathbf{a}) = 0. \end{aligned}$$

Thus we obtain  $(rU' + U)' = 0$ , and subsequently potential 5.17 (up to the negligible additive constant). Therefore, all of the three Kepler's laws have a group theoretical nature. This is demonstrated in Table 5.1 for two-body problem (elliptic orbits) with the Lagrangian  $L = (m/2)\dot{\mathbf{x}}^2 - \alpha/r$ .

**5.3. IS THE ANOMALY IN THE MOTION OF MERCURY COMPATIBLE WITH THE HUYGENS PRINCIPLE?**

The Einstein's explanation of the Mercury perihelion's motion rests on the assumption that the space-time near the Sun has the Schwarzschild metric. This metric was first found by Einstein [1915] with an accuracy up to the second-order approximation. However, the transition from the plain space-time of Minkowski to the Schwarzschild metric violates the Huygens principle and hence gives rise to the diffusion of acoustic, light, etc. waves.

This is due to the fact that the Minkowski space belongs to the family of Riemannian spaces with a nontrivial conformal group whereas the Schwarzschild space has a trivial conformal group and does not satisfy the Huygens principle (see, Chapter 4 of this volume or Ibragimov [1983]).

Thus, in connection with theoretical explanation of the anomalous motion of planets a new problem arises, that can be formulated as the following alternative:

1. An explanation via the transition to the Schwarzschild metric correctly accounts the nature of phenomenon in the necessary approximation. Then the Huygens principle is violated, and hence acoustic, light, etc. signals diffuse. The level of the diffusions are to be evaluated to be sure that the latter can be detected.
2. In the real world the Huygens principle is valid. Then an explanation of the motion of Mercury's perihelion is to be found such that it does not contradict the Huygens principle. This will demand a rigorous physical analysis of equations of motion in a curved space-time with a nontrivial conformal group. The problem is simplified due to the fact that the complete description of a family of all such spaces is known: every space-time with a nontrivial conformal group can be reduced, by the choice of a suitable coordinate system, to the plain wave metric

$$ds^2 = e^{\mu(x)} \left[ (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - 2f(x^1 - x^0) dx^2 dx^3 - g(x^1 - x^0)(dx^3)^2 \right], \quad (5.20)$$

where  $f$  and  $g$  are functions of one variable  $x^1 - x^0$  satisfying the hyperbolicity condition  $g - f^2 > 0$  and  $\mu$  is an arbitrary function of  $x$ .

## 5.4. PECULIARITIES OF GROUP MODELING IN THE DE SITTER UNIVERSE

We discuss here (Ibragimov [1993]) the following two peculiarities concerning the transition from the Minkowski geometry to the de Sitter universe (space-time of constant curvature).

The first of them is related to the possibility of treating the de Sitter group, in the context of the theory of approximate groups, as a perturbation of the Poincaré group by introducing a small constant curvature. In fact, according to cosmological data, the curvature of our Universe is so small ( $\sim 10^{-54} \text{ cm}^{-2}$ ) that it is sufficient to calculate first order perturbations only. Then Lie equations are readily solved and yield approximate representation of the de Sitter group. This is the way to simplify the theory.

The second peculiarity is the utilization of a special complex transformation. In case of Minkowski space, it reduces to a trivial conformal transforma-

tion, admitted by the Dirac equation with a zero mass, and does not play an essential role. However, in our approach, it becomes a nontrivial equivalence transformation on the set of spaces of constant curvature, and its addition to the de Sitter group integrates all three possible types of spaces with constant curvature: elliptic, hyperbolic (Lobatchevski space), and parabolic (Minkowski space as the limiting case of zero curvature).

#### 5.4.1. APPROXIMATE REPRESENTATION OF THE DE SITTER GROUP

Metric of the de Sitter space has the form:

$$ds^2 = \left(1 + \frac{K}{4}\rho^2\right)^{-2} (c^2 dt^2 - dx^2 - dy^2 - dz^2), \quad K = \text{const}, \quad (5.21)$$

where

$$\rho^2 = r^2 - c^2 t^2, \quad r^2 = x^2 + y^2 + z^2. \quad (5.22)$$

In the usual notation  $(x^1, x^2, x^3, x^4) = (x, y, z, ict)$ ,  $ds = i d\tau$  we have

$$d\tau^2 = \left(1 + \frac{K}{4}\sigma^2\right)^{-2} \sum_{\mu=1}^4 (dx^\mu)^2, \quad \sigma^2 = \sum_{\mu=1}^4 (x^\mu)^2. \quad (5.23)$$

The group of isometric motions in the de Sitter space is called the de Sitter group. It differs from the Poincaré group in that trivial translations of coordinates  $x^\mu$  are replaced by more complicated transformations, the so-called "generalized translations." For example, the generalized translation along the  $x^1$  axis is generated by

$$\begin{aligned} X_1 = & \left(1 + \frac{K}{4}[(x^1)^2 - (x^2)^2 - (x^3)^2 - (x^4)^2]\right) \frac{\partial}{\partial x^1} \\ & + \frac{K}{2}x^1 \left(x^2 \frac{\partial}{\partial x^2} + x^3 \frac{\partial}{\partial x^3} + x^4 \frac{\partial}{\partial x^4}\right) \end{aligned} \quad (5.24)$$

and has the form ( $a$  is a group parameter):

$$\bar{x}^1 = 2 \frac{x^1 \cos(a\sqrt{K}) + \left(1 - \frac{K}{4}\sigma^2\right) \frac{1}{\sqrt{K}} \sin(a\sqrt{K})}{1 + \frac{K}{4}\sigma^2 + \left(1 - \frac{K}{4}\sigma^2\right) \cos(a\sqrt{K}) - x^1 \sqrt{K} \sin(a\sqrt{K})}; \quad (5.25)$$

$$\bar{x}^l = 2 \frac{x^l}{1 + \frac{K}{4}\sigma^2 + \left(1 - \frac{K}{4}\sigma^2\right) \cos(a\sqrt{K}) - x^1 \sqrt{K} \sin(a\sqrt{K})}; \quad l = 2, 3, 4.$$

When the constant curvature  $K$  is small, one can utilize the approximate group theory (Baikov, Gazizov, and Ibragimov [1988]). The approximate Lie equation for Operator 5.24 is easily solved. As a result, one obtains the following simple approximate representation of Transformations 5.25 with the small parameter  $K$  (for details of calculations see Ibragimov [1990]):

$$\begin{aligned}\bar{x}^1 = x^1 + a + \frac{K}{4} \left\{ \left[ (x^1)^2 - (x^2)^2 - (x^3)^2 - (x^4)^2 \right] a \right. \\ \left. + x^1 a^2 + \frac{1}{3} a^3 \right\} + o(K),\end{aligned}\quad (5.26)$$

$$\bar{x}^l = x^l + \frac{K}{4} x^l (2ax^1 + a^2) + o(K), \quad l = 2, 3, 4.$$

#### 5.4.2. THE KEPLER PROBLEM IN THE DE SITTER SPACE

The free motion of a particle in the de Sitter space is described by the Lagrangian function

$$L = -mc^2\theta^{-1} \sqrt{1 - \beta^2}, \quad (5.27)$$

where

$$\theta = \left( 1 + \frac{K}{4} (r^2 - c^2 t^2) \right), \quad \beta^2 = \frac{v^2}{c^2}$$

and  $m$  and  $v$  are mass and velocity of a particle; we shall write  $\mathbf{x} = (x^1, x^2, x^3)$ , hence  $\mathbf{v} = d\mathbf{x}/dt$ .

Using this familiar formula, let us find the Lagrangian formulation of the Kepler problem in the de Sitter space. Heuristic reasoning for that is the invariance of the classical Kepler problem under rotations and time translations together with the known limiting expression 5.17 of the potential at  $K = 0$ ,  $\beta^2 \rightarrow 0$ , and Formula 5.27. Therefore we look for the Lagrangian of the form

$$L = -mc^2\theta^{-1} \sqrt{1 - \beta^2} + \frac{\alpha}{r} \theta^m (1 - \beta^2)^n, \quad m, n = \text{const.} \quad (5.28)$$

The action integral  $\int L dt$  with this Lagrangian is invariant with respect to rotations. Therefore the only additional condition is the invariance with respect to generalized translations in time with the generator (cf. Operator 5.24)

$$X_4 = \frac{1}{c^2} \left[ 1 - \frac{K}{4} (c^2 t^2 + r^2) \right] \frac{\partial}{\partial t} - \frac{K}{2} t \sum_{l=1}^3 x^l \frac{\partial}{\partial x^l}. \quad (5.29)$$

**Theorem 5.3.** (Ibragimov [1993]). The action integral  $\int L dt$  with the Lagrangian 5.28 is invariant under the group with Generator 5.29 if and only if  $m = 0$  and  $n = 1/2$ , i.e.,

$$L = -mc^2\theta^{-1} \sqrt{1 - \beta^2} + \frac{\alpha}{r} \sqrt{1 - \beta^2}. \quad (5.30)$$

**Proof.** The required invariance condition has the form (Ibragimov [1983])

$$[\tilde{X}_4 + D_t(\xi)]L = 0, \quad (5.31)$$

where  $\tilde{X}_4$  is the prolongation of the Operator 5.29 to  $\mathbf{v}$ , and  $\xi$  is a coordinate of the operator at  $\partial/\partial t$ . Calculations yield

$$[\tilde{X}_4 + D_t(\xi)]L = \frac{\alpha K}{2r} \theta^m (1 - \beta^2)^n \left[ (1 - 2n) \frac{\mathbf{x} \cdot \mathbf{v}}{c^2} + mt \right]. \quad (5.32)$$

Therefore, the statement of the theorem follows from Equation 5.31.

**Remark.** The analogous theorem, concerning the uniqueness of the invariant Lagrangian, is not valid in Minkowski space. Indeed, if  $K = 0$  it follows from Equation 5.32 that Equation 5.31 is an identity for Lagrangian 5.28 with an arbitrary  $n$ .

### 5.4.3. SPLITTING OF NEUTRINOS BY THE CURVATURE

Spinor analysis in a curved space was developed by different authors from various points of view. A good presentation of a necessary apparatus and general references are to be found in the review paper by Brill and Wheeler [1957]. In accordance with Section 2 of this review the Dirac equation in the de Sitter metric 5.23 can be written as

$$\left(1 + \frac{K}{4} \sigma^2\right) \gamma^\mu \frac{\partial \psi}{\partial x^\mu} - \frac{3}{4} K (\mathbf{x} \cdot \boldsymbol{\gamma}) \psi + m \psi = 0, \quad m = \text{const}, \quad (5.33)$$

where  $\gamma^\mu$  ( $\mu = 1, \dots, 4$ ) are usual Dirac matrices in the Minkowski space, and  $(\mathbf{x} \cdot \boldsymbol{\gamma})$  denotes a four-dimensional scalar product:

$$(\mathbf{x} \cdot \boldsymbol{\gamma}) = \sum_{\mu=1}^4 x^\mu \gamma^\mu.$$

Here we will be interested only in neutrino equations ( $m = 0$ ). Then, in linear (with respect to  $K$ ) approximation, Equation 5.33 becomes:

$$\gamma^\mu \frac{\partial \psi}{\partial x^\mu} - \frac{3}{4} K (\mathbf{x} \cdot \boldsymbol{\gamma}) \psi = 0. \quad (5.34)$$

Equation 5.33 admits the de Sitter group, the action of which upon wave function  $\psi$  is defined as follows. Let us write the infinitesimal transformations of de Sitter group in the form  $\delta \mathbf{x} = a \xi$ . Then  $\delta \psi = a S \psi$ , where

$$S = \frac{1}{8} \sum_{\mu, \nu=1}^4 \frac{\partial \xi^\mu}{\partial x^\nu} (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu - 3 \delta_{\mu\nu}) + \frac{3}{4} K \left( 1 + \frac{K}{4} \sigma^2 \right)^{-1} (\mathbf{x} \cdot \xi). \quad (5.35)$$

Now we turn to Equation 5.34. It can be reduced (in the linear approximation with respect to  $K$ ) to the usual Dirac equation

$$\gamma^\mu \frac{\partial \varphi}{\partial x^\mu} = 0 \quad (5.36)$$

by the substitution

$$\varphi = \psi(x) \exp \left[ \frac{1}{2} \left( 1 + \frac{K}{4} \sigma^2 \right)^{-3} \right]. \quad (5.37)$$

Equation 5.36 is invariant under the transformation

$$\bar{x}^\mu = i x^\mu, \quad i = \sqrt{-1}. \quad (5.38)$$

Therefore, one can choose, instead of Equation 5.37, the different representation for the function  $\varphi$ , viz.

$$\varphi = \chi(\bar{x}) \exp \left[ \frac{1}{2} \left( 1 + \frac{K}{4} \sigma^2 \right)^{-3} \right]. \quad (5.37')$$

Then Equation 5.36 yields (after eliminating the bar upon  $\bar{x}$ ):

$$\gamma^\mu \frac{\partial \chi}{\partial x^\mu} + \frac{3}{4} K (\mathbf{x} \cdot \gamma) \chi = 0. \quad (5.39)$$

Equation 5.39 coincides with Equation 5.34 in the de Sitter space with the curvature  $(-K)$ .

Transformation 5.38 converts time-like intervals in Minkowski space to space-like intervals, and vice versa. In the case of de Sitter spaces the same happens along with the simultaneous change of the sign of the curvature. Indeed, the Interval 5.23 being furnished with a subscript  $K$ , Transformation 5.38 yields

$$ds_{(K)}^2 = -d\bar{s}_{(-K)}^2. \quad (5.40)$$

Formulas 5.37 and 5.37' can be interpreted as the "splitting" of a neutrino into two neutrinos, described, respectively, by Equations 5.34 and 5.39. They are different only when  $K \neq 0$ . The system of Equations 5.34 and 5.39 admits the approximate group defined by the infinitesimal transformations

$$\delta \mathbf{x} = a \boldsymbol{\xi}, \quad \delta \psi = a S \psi, \quad \delta \chi = a T \chi, \quad (5.41)$$

where the vector  $\boldsymbol{\xi} = (\xi^1, \xi^2, \xi^3, \xi^4)$  belongs to the 15-dimensional Lie algebra of the group of conformal transformations of Minkowski space (see, e.g., Ibragimov [1983]), and matrices  $S$  and  $T$  are given by

$$S = \frac{1}{8} \sum_{\mu, \nu=1}^4 \frac{\partial \xi^\mu}{\partial x^\nu} (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu - 3\delta_{\mu\nu}) + \frac{3}{4} K(\mathbf{x} \cdot \boldsymbol{\xi}),$$

$$T = \frac{1}{8} \sum_{\mu, \nu=1}^4 \frac{\partial \xi^\mu}{\partial x^\nu} (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu - 3\delta_{\mu\nu}) - \frac{3}{4} K(\mathbf{x} \cdot \boldsymbol{\xi}). \quad (5.42)$$

## 5.5. DYNAMIC PHENOMENA IN SPACE PLASMA

Applications of Lie groups to the equilibrium theory of cylindrically symmetric magnetic flux tubes are considered here. These results are due to Milovanov and Zelenyi [1991]–[1993b] and Milovahov [1993a]–[1993c].

The self-consistent equilibrium of the tubes is governed by the dimensionless Vlasov–Maxwell equations

$$-\frac{\partial}{\partial \rho} \left( \frac{1}{\rho} \frac{\partial u}{\partial \rho} \right) = \rho \nu(u + v), \quad (5.43)$$

$$-\frac{\partial}{\partial \rho} \left( \frac{1}{\rho} \frac{\partial v}{\partial \rho} \right) = \rho \nu(u + v), \quad (5.44)$$

where the plasma distribution function is assumed to depend on a linear combination of constants of the motion. In Equations 5.43 and 5.44,  $\rho$  is the dimensionless polar radius,  $\nu \equiv \nu(u + v)$  is the dimensionless plasma density, and  $u$  and  $v$  are the components of the vector potential. Equations 5.43 and 5.44 are simplified by transforming to the new variable  $x = \rho^2$ :

$$4 \frac{d}{dx} \frac{du}{dx} = -\nu(u + v), \quad (5.45)$$

$$4 \frac{d}{dx} \frac{dv}{d \ln x} = -\nu(u + v). \quad (5.46)$$

Subtracting the second equation from the first and integrating over  $x$ , we find

$$\frac{du}{dx} = x \frac{dv}{d \ln x}. \quad (5.47)$$

Here, the integration constant is taken to be zero which implies a relation between the components of the magnetic field of the tube. Differentiating Equation 5.45 with respect to  $x$  and using Equation 5.47, we arrive at the third-order nonlinear differential equation

$$u_{xxx} = \frac{x+1}{x} u_x \mu(u), \quad (5.48)$$

where the function  $\mu(u)$  is defined by

$$-4 \int^{u+v} \mu(w) dw = v(u+v). \quad (5.49)$$

Analytical solutions to Equation 5.48 exist in both limiting cases,  $x \rightarrow 0$  and  $x \rightarrow \infty$ , only for the functions  $\mu(u) = \pm \exp u$  and  $\mu(u) = \pm u^p$ , where  $p$  is any real number. In fact, when  $\mu(u) = \pm \exp u$ , one finds a nontrivial one-dimensional Lie algebra with the basis

$$X_1 = x \frac{\partial}{\partial x} - \frac{\partial}{\partial u} \quad (5.50)$$

in the limiting case  $x \rightarrow 0$ , and a two-dimensional Lie algebra with the basis

$$X_1 = x \frac{\partial}{\partial x} - 2 \frac{\partial}{\partial u}, \quad X_2 = \frac{\partial}{\partial x}, \quad (5.51)$$

in the limiting case  $x \rightarrow \infty$ . Similarly, when  $\mu(u) = \pm u^p$ , we have

$$X_1 = x \frac{\partial}{\partial x} - \frac{u}{p} \frac{\partial}{\partial u} \quad (5.52)$$

in the case  $x \rightarrow 0$ , and

$$X_1 = x \frac{\partial}{\partial x} - 2 \frac{u}{p} \frac{\partial}{\partial u}, \quad X_2 = \frac{\partial}{\partial x}, \quad (5.53)$$

in the case  $x \rightarrow \infty$ . Using Equations 5.49–5.53, one can obtain particular solutions for the magnetic fields and plasma density in the tubes which play an important role when considering dynamic evolution of large-scale plasma configurations in the solar photosphere and interplanetary medium.

Another possible application of Lie groups concern spontaneous polymerization of magnetic field fluctuations during the radiation epoch ( $t = 10^8 - 10^{13}$  s after the big bang) of the evolution of the Universe which is governed by the nonlinear kinetic equation of the form

$$\frac{\partial p}{\partial t} = \Delta p + Q(x)p^\gamma. \quad (5.54)$$

In equation 5.54  $x \equiv |\mathbf{x}|$ , the function  $p = p(\mathbf{x}, t)$  is the probability density to find a fluctuation  $\mu^+$ , say, at a point  $\mathbf{x}$  at a time  $t$ , meanwhile  $Q(x) \propto x^{D-3}$  shows the average probability of breakdown of symmetry in the creation-annihilation reaction  $0 \rightleftharpoons \mu^+ \mu^-$ . The diffusion term  $\Delta p$  is defined as (e.g., see O'Shaughnessy and Procaccia [1985])

$$\Delta p = \frac{1}{x^{D-1}} \frac{\partial}{\partial x} \left( K(x) x^{D-1} \frac{\partial p}{\partial x} \right), \quad (5.55)$$

which is a generalization of the Euclidean diffusion operator on a fractal of dimension  $D$ . Here the function  $K(x)$  is the diffusion coefficient:

$$K(x) = Kx^{-\sigma}, \quad (5.56)$$

where  $K$  is the diffusion constant, and  $\sigma$  is the index of anomalous diffusion (Gefen, Aharony and Alexander [1983]). Note, that for Euclidean geometries,  $\sigma \equiv 0$ . Then, let us define the new function  $u(x, t) = x^{-\beta/2} p(x, t)$ , where  $\beta = 1 + \delta - D = \text{const.}$  Combining Equations 5.44–5.56 after some algebra, we obtain

$$u_{xx} = \frac{\beta(\beta + 2)}{4} \frac{u}{x^2} + x^\sigma u_t - x^\theta u^\gamma. \quad (5.57)$$

In Equation 5.57,  $\theta = \sigma + D - 3 - \beta(1 - \gamma)/2$ ; moreover, we assume  $\gamma \neq 1$  and  $\sigma > -2$ . Without lack of generality, we also set  $K = 1$ . Comprehensive analysis shows that Equation 5.57 admits the two-dimensional Lie-algebra spanned by

$$X_1 = t \frac{\partial}{\partial t} + \frac{x}{\sigma + 2} \frac{\partial}{\partial x} + \omega u \frac{\partial}{\partial u}, \quad (5.58)$$

$$X_2 = \frac{\partial}{\partial t}, \quad (5.59)$$

where  $\omega = (\theta + 2)/(\sigma + 2)(1 - \gamma)$ . The infinitesimal operator  $X_1$  defined by Equation 5.58, yields the invariant solution to Equation 5.58:

$$P = x^{(\beta/2)} u = Cx^{-(\sigma+D-1)/(\gamma-1)}, \quad (5.60)$$

where constant  $C$  is defined by the relation

$$4C^{\nu-1} = \beta^2 + 2\beta + 4\omega(\sigma + 2)[(\omega - 1)(\sigma + 2) + (\sigma + 1)]. \quad (5.61)$$

It is shown in Milovanov [1993a, b], that the invariant solution 5.60 can be used for the direct calculation of the fractal dimension of magnetic polymers in early Universe. The results obtained account for the present large-scale distribution of galaxies and galaxy clusters at  $\sim 10 h^{-1}$  Mpc scales where  $h = H_0/100$ , and  $H_0$  is the Hubble constant. (For details, see, e.g., Peebles [1980] and Bahcall and West [1992].)

We now observe that the nonlinear kinetic equation 5.60 is the generalization of the linear diffusion equation

$$\frac{\partial p}{\partial t} = \Delta p, \quad (5.62)$$

which is of great theoretical and conceptual interest in different branches of physics (e.g., see O'Shaughnessy and Procaccia [1985], Clement, Kopelman and Sander [1991], and Lawrence [1991]). Equation 5.62 with the diffusion term 5.55 describes fractional Brownian motion of the hot stars in globular clusters and anomalous diffusion of magnetic flux elements in the solar photosphere. Calculation of the admissible Lie group of Equation 5.62 enables one to find basic analytical relations between the fractal dimension  $D$  and the index of anomalous diffusion  $\sigma$  when considering self-similar behavior of dynamic systems in fractal geometries (for details, see Milovanov [1993c]). Here we present the admissible Lie group of Equation 5.62 with the diffusion term 5.55 in explicit form.

First of all, when the index of anomalous diffusion  $\sigma$  is arbitrary, Equation 5.62 admits the four-dimensional Lie algebra with the basis

$$\begin{aligned} X_1 &= t^2 \frac{\partial}{\partial t} + \frac{2tx}{\sigma + 2} \frac{\partial}{\partial x} - \left[ \frac{x^{\sigma+2}}{(\sigma + 2)^2} + \frac{Dt}{\sigma + 2} \right] u \frac{\partial}{\partial u}, \\ X_2 &= t \frac{\partial}{\partial t} + \frac{x}{\sigma + 2} \frac{\partial}{\partial x}, \\ X_3 &= \frac{\partial}{\partial t}, \quad X_4 = u \frac{\partial}{\partial u}. \end{aligned} \quad (5.63)$$

When  $\gamma = 2D - 2$ , we find two additional operators:

$$\begin{aligned} X_5 &= tx^{-\sigma/2} \frac{\partial}{\partial x} - \frac{x^{(\sigma+2)/2}}{\sigma + 2} u \frac{\partial}{\partial u}, \\ X_6 &= x^{-\sigma/2} \frac{\partial}{\partial x}, \end{aligned} \quad (5.64)$$

which form, together with 5.63, a six-dimensional Lie algebra. Finally, when  $\sigma = (2D/3) - 2$ , the additional operators become

$$\begin{aligned} X_5 &= tx^{-\sigma/2} \frac{\partial}{\partial x} - \left[ \frac{\sigma+2}{2} tx^{-(\sigma+2)/2} + \frac{x^{(\sigma+2)/2}}{\sigma+2} \right] u \frac{\partial}{\partial u}, \\ X_6 &= x^{-\sigma/2} \frac{\partial}{\partial x} - \frac{\sigma+2}{2} x^{-(\sigma+2)/2} u \frac{\partial}{\partial u}. \end{aligned} \quad (5.65)$$

The generalized invariant solution to Equation 5.62 is given by:

$$P = \frac{\sigma+2}{D\Gamma(D/(\sigma+2))} \left[ \frac{1}{(\sigma+2)^2 t} \right]^{D/(\sigma+2)} \exp \left[ -\frac{x^{\sigma+2}}{(\sigma+2)^2 t} \right]. \quad (5.66)$$

# 6

## Utilization of Vessiot–Guldberg–Lie Algebra for Integration of Nonlinear Equations

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This chapter is aimed to illustrate a utility of the method of canonical variables (see [H1], Chapter 2) combined with the result due to Vessiot [1893], Guldberg [1893], and Lie [1893] on nonlinear ordinary differential equations possessing fundamental systems of solutions.

### 6.1. DEFINITION OF A FUNDAMENTAL SYSTEM OF SOLUTIONS

Ordinary differential equations

$$\frac{dx^i}{dt} = F_i(t, x^1, \dots, x^n), \quad i = 1, \dots, n \quad (6.1)$$

are said to possess a fundamental system of solutions if their general solution can be expressed in terms of a finite number  $m$  of arbitrary particular solutions

$$x_k = (x_k^1, \dots, x_k^n), \quad k = 1, \dots, m \quad (6.2)$$

by formulas

$$x^i = \varphi^i(x_1, \dots, x_m, C_1, \dots, C_n), \quad (6.3)$$

where  $C_1, \dots, C_n$  are arbitrary constants.

Particular solutions 6.2 are referred to as fundamental system of solutions for Equations 6.1. It is required that the form of functions  $\varphi^i$  does not depend on the choice of specific particular solutions 6.2. This, however, does not exclude, for a given system of Equations 6.1, the possibility to have several distinct representations 6.3 of a general solution as well as different numbers  $m$  of necessary particular solutions.

## 6.2. VESSIOT–GULDBERG–LIE ALGEBRA

The question of which of Equation 6.1 possess a fundamental system of solutions has no direct relation to symmetry groups. However, Vessiot [1893], Guldberg [1893], and Lie [1893] revealed that this question can be solved in terms of Lie algebras. In fact, Lie found the general form of equations with fundamental systems of solutions and proved the following main statement.

**Theorem 6.1.** Equations 6.1 possess the fundamental system of solutions if and only if they have a special form

$$\frac{dx^i}{dt} = T_1(t)\xi_1^i(x) + \dots + T_r(t)\xi_r^i(x) \quad (6.4)$$

so that operators

$$X_\alpha = \xi_\alpha^i(x) \frac{\partial}{\partial x^i}, \quad \alpha = 1, \dots, r \quad (6.5)$$

span an  $r$ -dimensional Lie algebra  $L_r$ . The number  $m$  of necessary particular solutions 6.2 is estimated by

$$nm \geq r. \quad (6.6)$$

The proof of this theorem with detailed preliminary discussions and examples are to be found in Lie [1893], Chapter 24. The Lie algebra  $L_r$  that results from Theorem 6.1 will be referred to as Vessiot–Guldberg–Lie algebra.

### 6.3. APPLICATION TO NONLINEAR EQUATIONS OF LASER SYSTEMS

#### 6.3.1. ADAPTATION OF THE GALILEAN GROUP

The phenomena of the wave front correction for optical radiations in laser systems are simulated by nonlinear equations known as the system of phase-conjugated reflection (wave front reversal) equations (see Zel'dovitch, Pilipetskii, and Shkunov [1985] and Bespalov and Pasmannik [1986]; see also Section 17.12 of this volume). Here, we shall consider a simplified (by choosing the medium parameters) variant of this system and restrict the discussion to the stationary case. Then the equations are written

$$\begin{aligned} \left( \frac{\partial}{\partial z} - i\Delta \right) E_1 &= |E_2|^2 E_1, \\ \left( \frac{\partial}{\partial z} + i\Delta \right) E_2 &= |E_1|^2 E_2, \end{aligned} \quad (6.7)$$

where  $\Delta$  is the Laplace operator on the  $(x, y)$  plane,  $E_1$  and  $E_2$  are complex amplitudes of the incident and phase conjugated (amplified) light waves, respectively.

Equations 6.7 are evidently invariant under the translations along the  $x$ ,  $y$ , and  $z$  axis, rotations on the  $(x, y)$  plane and appropriate dilations of dependent and independent variables. Now, we shall find an additional symmetry group using the analogy of the left-hand sides of Equations 6.7 and the heat equation

$$u_t - u_{xx} = 0. \quad (6.8)$$

Equation 6.8 admits the Lie algebra  $L_6$  spanned by

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, & X_2 &= \frac{\partial}{\partial x}, & X_3 &= 2t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x}, & X_4 &= u \frac{\partial}{\partial u}, \\ X_5 &= 2t \frac{\partial}{\partial x} - xu \frac{\partial}{\partial u}, & X_6 &= t^2 \frac{\partial}{\partial t} + tx \frac{\partial}{\partial x} - \frac{1}{2}(x^2 + 2t)u \frac{\partial}{\partial u} \end{aligned} \quad (6.9)$$

and the infinite dimensional algebra of operators  $X_\mu = \mu(t, x) \partial / \partial u$  where  $\mu(t, x)$  is an arbitrary solution of Equation 6.8 (see [H1]).

Turning to Equations 6.7, we shall look for a symmetry generator as the following modification of the generator  $X_5$  of the Galilean group:

$$X = 2z \frac{\partial}{\partial x} + \sum_{\alpha=1}^2 \left( f_\alpha(x, z) E_\alpha \frac{\partial}{\partial E_\alpha} + g_\alpha(x, z) E_\alpha^* \frac{\partial}{\partial E_\alpha^*} \right),$$

where  $E_\alpha^*$  and  $E_\alpha$  are complex conjugate quantities. The choice of the operator in this specific form simplifies the solution of the determining

equations and yields

$$X = 2z \frac{\partial}{\partial x} + ix \left( E_1 \frac{\partial}{\partial E_1} - E_2 \frac{\partial}{\partial E_2} - E_1^* \frac{\partial}{\partial E_1^*} + E_2^* \frac{\partial}{\partial E_2^*} \right). \quad (6.10)$$

### 6.3.2. AN INVARIANT SOLUTION

Consider the two-dimensional abelian Lie algebra spanned by Operator 6.10 and the Operator  $\partial/\partial y$ . The basis of invariants is given by

$$z, \quad u_1 = E_1 e^{-ix^2/(4z)}, \quad u_2 = E_2 e^{ix^2/(4z)},$$

and the complex conjugates for  $u_1$  and  $u_2$ . Hence, we have the following general form of the invariant solutions:

$$E_1 = u_1(z) e^{ix^2/(4z)}, \quad E_2 = u_2(z) e^{-ix^2/(4z)}. \quad (6.11)$$

For the simplicity sake, we consider the case of real functions  $u_1$  and  $u_2$ . Then the substitution of Expressions 6.11 in Equations 6.7 yields

$$\frac{du_1}{dz} = u_2^2 u_1 - \frac{u_1}{2z}, \quad \frac{du_2}{dz} = u_1^2 u_2 - \frac{u_2}{2z}. \quad (6.12)$$

### 6.3.3. UTILITY OF CANONICAL VARIABLES

Equations 6.12 have the form of Equations 6.4 with the coefficients

$$T_1(z) = 1, \quad T_2(z) = -\frac{1}{2z}$$

and with Operators 6.5 given by

$$X_1 = u_2^2 u_1 \frac{\partial}{\partial u_1} + u_1^2 u_2 \frac{\partial}{\partial u_2}, \quad X_2 = u_1 \frac{\partial}{\partial u_1} + u_2 \frac{\partial}{\partial u_2}. \quad (6.13)$$

We have

$$[X_1, X_2] = -2X_1, \quad X_1 \vee X_2 = u_1 u_2 (u_2^2 - u_1^2) \neq 0.$$

Therefore Operators 6.13 span a two-dimensional Lie algebra of type III given in [H1], Section 2.2.1 and hence can be transformed to the canonical forms. The use of canonical variables transforms Equations 6.12 to an integrable form.

We find the canonical variables by solving the equations

$$X_1(v_1) = 0, \quad X_2(v_2) = 1,$$

and obtain

$$v_1 = u_1^2 - u_2^2, \quad v_2 = \frac{\ln u_2 - \ln u_1}{u_1^2 - u_2^2}. \quad (6.14)$$

Then Operators 6.13 become

$$X_1 = \frac{\partial}{\partial v_2}, \quad X_2 = 2 \left( v_1 \frac{\partial}{\partial v_1} + v_2 \frac{\partial}{\partial v_2} \right). \quad (6.13')$$

Hence, in Variables 6.14, Equations 6.12 reduce to the following:

$$\frac{\partial v_1}{\partial z} = -\frac{v_1}{z}, \quad \frac{dv_2}{dz} = 1 + \frac{v_2}{z}. \quad (6.12')$$

The solution of Equations 6.12' is given by

$$v_1 = -\frac{C_1}{z}, \quad v_2 = C_2 z + z \ln z. \quad (6.15)$$

Now we reverse Formulas (6.14):

$$u_1 = \sqrt{\frac{v_1}{1 - e^{2v_1v_2}}}, \quad u_2 = \sqrt{\frac{v_1}{e^{-2v_1v_2} - 1}} \quad (6.14')$$

and substitute herein solutions 6.15. It follows that the general solution of Equations 6.12 is given by

$$u_1 = \sqrt{\frac{k}{z(1 - \zeta^2)}}, \quad u_2 = \zeta \sqrt{\frac{k}{z(1 - \zeta^2)}} \quad (6.16)$$

where  $\zeta = Cz^k$ ,  $C, k = \text{const}$ . Substituting this in Formulas 6.11 one obtains the invariant solution of the System 6.7:

$$E_1 = \sqrt{\frac{k}{z(1 - \zeta^2)}} e^{ix^2/(4z)}, \quad E_2 = \zeta \sqrt{\frac{k}{z(1 - \zeta^2)}} e^{ix^2/(4z)},$$

where  $\zeta = Cz^k$ , and  $C, k$  are arbitrary constants.

## *B. Body of Results*

# Symmetry Groups and Fundamental Solutions for Linear Equations of Mathematical Physics

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## 7.1. LAPLACE EQUATION

$$\Delta_n u = 0, \quad n \geq 3, \quad (7.1)$$

where  $\Delta_n$  is the Laplacian in  $\mathbb{R}^n$ ,

$$\Delta_n = \frac{\partial^2}{(\partial x^1)^2} + \cdots + \frac{\partial^2}{(\partial x^n)^2}.$$

The symmetry algebra  $L_r, r = \frac{1}{2}(n+1)(n+2)+1$ , is spanned by (see, e.g., Ibragimov [1972] and [1983], Section 10.2)

$$\begin{aligned} X_i &= \frac{\partial}{\partial x^i}, & X_{ij} &= x^j \frac{\partial}{\partial x^i} - x^i \frac{\partial}{\partial x^j}, \\ Y_i &= (2x^i x^j - |x|^2 \delta^{ij}) \frac{\partial}{\partial x^j} - (n-2)x^i u \frac{\partial}{\partial u}, \end{aligned} \quad (7.2)$$

$$Z_1 = x^i \frac{\partial}{\partial x^i}, \quad Z_2 = u \frac{\partial}{\partial u}, \quad (i, j = 1, \dots, n),$$

where  $|x|^2 = (x^1)^2 + \dots + (x^n)^2$ .

The equation for fundamental solutions,

$$\Delta_n u = \delta(x), \quad (7.3)$$

where  $\delta$  is the Dirac measure, admits the subalgebra of  $L_r$  spanned by

$$X_{ij}, \quad Z = Z_1 - (n-2)Z_2, \quad Y_i.$$

The fundamental solution

$$u = \frac{1}{(2-n)\omega_n} |x|^{2-n}, \quad (7.4)$$

where  $\omega_n = 2\pi^{n/2}/\Gamma(n/2)$  is the area of the unit sphere in  $\mathbb{R}^n$ , is determined by the requirement to be an invariant solution under the group generated by  $X_{ij}$  and  $Z$  (Ibragimov [1989], [1992]). In addition, this solution is invariant under the conformal transformations generated by  $Y_i$ .

## 7.2. HEAT EQUATION

$$u_t - \Delta_n u = 0. \quad (7.5)$$

The symmetry algebra  $L_r$ ,  $r = \frac{1}{2}(n+1)(n+2) + 3$ , is spanned by (Lie [1881] and Goff [1927])

$$X_0 = \frac{\partial}{\partial t}, \quad X_i = \frac{\partial}{\partial x^i}, \quad X_{ij} = x^j \frac{\partial}{\partial x^i} - x^i \frac{\partial}{\partial x^j},$$

$$X_{0i} = 2t \frac{\partial}{\partial x^i} - x^i u \frac{\partial}{\partial u}, \quad Z_1 = 2t \frac{\partial}{\partial t} + x^i \frac{\partial}{\partial x^i}, \quad Z_2 = u \frac{\partial}{\partial u},$$

$$Y = t^2 \frac{\partial}{\partial t} + tx^i \frac{\partial}{\partial x^i} - \frac{1}{4}(|x|^2 + 2nt)u \frac{\partial}{\partial u}, \quad (i, j = 1, \dots, n).$$

The equation for fundamental solutions

$$u_t - \Delta_n u = \delta(t, x) \quad (7.6)$$

admits the subalgebra of  $L_r$  spanned by

$$X_{ij}, \quad X_{0i}, \quad Z = Z_1 - nZ_2, \quad Y. \quad (7.7)$$

### The fundamental solution

$$u = \Theta(t)(2\sqrt{\pi t})^{-n} \exp\left(-\frac{|x|^2}{4t}\right), \quad (7.8)$$

$\Theta(t)$  being the Heaviside function, is obtained as an invariant solution under the group generated by  $X_{ij}$ ,  $X_{0i}$ , and  $Z$  (Ibragimov [1989], [1992]).

The solution of Cauchy's problem

$$u_t - \Delta_n u = f(t, x), \quad (7.9)$$

$$u|_{t=t_0} = u_0(x), \quad (7.10)$$

where  $f \in C^2$  ( $t \geq 0$ ),  $u_0(x) \in C(\mathbb{R}^n)$ , is given by Poisson's formula:

$$\begin{aligned} u(x, t) = & \int_0^1 \int_{\mathbb{R}^n} \frac{f(\xi, \tau)}{(\sqrt{\pi(t-\tau)})^n} \exp\left(-\frac{|x-\xi|^2}{4(t-\tau)}\right) d\xi d\tau \\ & + \frac{\Theta(t)}{(2\sqrt{\pi t})^n} \int_{\mathbb{R}^n} u_0(\xi) \exp\left(-\frac{|x-\xi|^2}{4t}\right) d\xi. \end{aligned} \quad (7.11)$$

### 7.3. WAVE EQUATION

$$u_{tt} - \Delta_n u = 0, \quad n \geq 2. \quad (7.12)$$

The symmetry algebra  $L_r$ ,  $r = \frac{1}{2}(n+2)(n+3) + 1$  is spanned by

$$\begin{aligned} X_0 &= \frac{\partial}{\partial t}, & X_i &= \frac{\partial}{\partial x^i}, & X_{ij} &= x^j \frac{\partial}{\partial x^i} - x^i \frac{\partial}{\partial x^j}, \\ X_{0i} &= t \frac{\partial}{\partial x^i} + x^i \frac{\partial}{\partial t}, & Z_1 &= t \frac{\partial}{\partial t} + x^i \frac{\partial}{\partial x^i}, & Z_2 &= u \frac{\partial}{\partial u}, \\ Y_0 &= (t^2 + |x|^2) \frac{\partial}{\partial t} + 2tx^i \frac{\partial}{\partial x^i} - (n-1)tu \frac{\partial}{\partial u}, \\ Y_i &= 2tx^i \frac{\partial}{\partial t} + [2x^i x^j + (t^2 - |x|^2) \delta^{ij}] \frac{\partial}{\partial x^j} - (n-1)x^i u \frac{\partial}{\partial u}, \\ &(i, j = 1, \dots, n). \end{aligned} \quad (7.13)$$

The equation for fundamental solutions,

$$u_{tt} - \Delta_n u = \delta(t, x) \quad (7.14)$$

admits the subalgebra of  $L_r$  spanned by

$$X_{ij}, X_{0i}, Z = Z_1 + (1 - n)Z_2, Y_0, Y_i. \quad (7.15)$$

The fundamental solution

$$u = \begin{cases} \frac{1}{2} \pi^{(1-n)/2} \theta(t) \delta^{(n-3)/2}(t^2 - |x|^2), & n \text{ odd} \\ \frac{1}{2} \pi^{-n/2} \theta(t) \left( \frac{\Theta(t^2 - |x|^2)}{\sqrt{t^2 - |x|^2}} \right)^{(n-2)/2}, & n \text{ even} \end{cases} \quad (7.16)$$

is a weak invariant solution with respect to the group generated by  $X_i$ ,  $X_{0i}$ , and  $Z$  (Berest [1991], [1993b], Ibragimov [1989], [1992]).

The solution of Cauchy's problem

$$u_{tt} - c^2 \Delta_n u = f(t, x), \quad c = \text{const},$$

$$u|_{t=t_0} = u_0(x), \quad \frac{\partial u}{\partial t} \Big|_{t=t_0} = u_1(x),$$

where  $f \in \mathbb{C}^1$  ( $t \geq 0$ ),  $u_0 \in \mathbb{C}^2(\mathbb{R}^n)$ ,  $u_1 \in \mathbb{C}^2(\mathbb{R}^n)$  for  $n = 1$ , and  $f \in \mathbb{C}^2$  ( $t \geq 0$ ),  $u_0 \in \mathbb{C}^3(\mathbb{R}^n)$ ,  $u_1 \in \mathbb{C}^2(\mathbb{R}^n)$  for  $n = 2, 3$  is given by D'Alembert [1746] for  $n = 1$ :

$$\begin{aligned} u(x, t) = & \frac{1}{2c} \int_0^t d\tau \int_{x-c(t-\tau)}^{x+c(t-\tau)} f(\xi, \tau) d\xi + \frac{1}{2c} \int_{x-ct}^{x+ct} u_1(\xi) d\xi \\ & + \frac{1}{2} [u_0(x+t) + u_0(x-t)], \end{aligned} \quad (7.17)$$

by Poisson [1819] for  $n = 2$ :

$$\begin{aligned} u(x, t) = & \frac{1}{2\pi c} \int_0^1 \int_{|\xi-x| < c(t-\tau)} \frac{f(\xi, \tau)}{\sqrt{c^2(t-\tau)^2 - |x-\xi|^2}} d\xi d\tau \\ & + \frac{1}{2\pi c^2} \int_{|\xi-x| < ct} \frac{u_1(\xi)}{\sqrt{c^2 t^2 - |x-\xi|^2}} d\xi \\ & + \frac{\partial}{\partial t} \left( \frac{1}{2\pi c} \int_{|\xi-x| < ct} \frac{u_0(\xi)}{\sqrt{c^2 t^2 - |x-\xi|^2}} d\xi \right), \end{aligned} \quad (7.18)$$

by Poisson [1819] and Kirchhoff [1882] for  $n = 3$  (see also Hadamard [1923], page 47 and Leray [1953], page 45):

$$u(x, t) = \frac{1}{4\pi c^2} \int_{|\xi-x|<ct} \frac{f(\xi, t - |x-\xi|/c)}{|x-\xi|} d\xi + \frac{1}{4\pi tc^2} \int_{|\xi-x|=ct} u_1(\xi) d\xi + \frac{\partial}{\partial t} \left\{ \frac{1}{4\pi tc^2} \int_{|\xi-x|=ct} u_0(\xi) d\xi \right\}. \quad (7.19)$$

For solution formulas in higher dimensions due to Tedone [1898], see, e.g., Courant [1962], Chapter VI, Section 12 and Ibragimov [1983], Section 13.4.

## 7.4. TRANSFER EQUATION

$$\frac{1}{v} u_t + s^i u_i + \gamma u = 0, \quad (7.20)$$

where  $u_i = \partial u / \partial x^i$ ,  $i = 1, \dots, n$ ;  $s^i, v, \gamma$  are arbitrary constants, and  $|s|^2 = 1$ . The symmetry algebra  $L_\infty$  is infinite dimensional and spanned by

$$X = \alpha(x, t) \frac{\partial}{\partial t} + [f^i(x - vst) + v\alpha(x, t)s^i] \frac{\partial}{\partial x^i} - (g(x - vst) + v\gamma\alpha(x, t))u \frac{\partial}{\partial u}, \quad (7.21)$$

where  $\alpha, f^i, g$  are arbitrary smooth functions.

The equation for fundamental solutions,

$$\frac{1}{v} u_t + s^i u_i + \gamma u = \delta(x, t), \quad (7.22)$$

admits the subalgebra of  $L_\infty$  spanned by Operator 7.21 provided that (Berest [1993b])

$$\alpha(0, 0) = 0, \quad f^i(0) = 0, \quad g(0) = \sum_{i=1}^n \frac{\partial f^i}{\partial x^i}(0). \quad (7.23)$$

The fundamental solution

$$u(x, t) = v\Theta(t)\exp(-\gamma vt)\delta(x - vst) \quad (7.24)$$

is a weak invariant solution of Equation 7.22 under the group generated by Operator 7.21 with Conditions 7.23.

## 7.5. CAUCHY-RIEMANN EQUATION

$$\frac{\partial u}{\partial \bar{z}} = 0, \quad (7.25)$$

where  $u = u(z, \bar{z})$ ,  $z = x + iy$ ,  $\bar{z} = x - iy$ , and

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

The symmetry algebra  $L_\infty$  is spanned by

$$X = \alpha(z) \frac{\partial}{\partial z} + \beta(z, \bar{z}) \frac{\partial}{\partial \bar{z}} + \eta(z) u \frac{\partial}{\partial u}, \quad (7.26)$$

where  $\alpha(z)$ ,  $\eta(z)$  are arbitrary analytic functions depending on the one complex variable  $z$ , and  $\beta(z, \bar{z})$  is a smooth function of the variables  $z$  and  $\bar{z}$ .

The equation for fundamental solutions,

$$\frac{\partial u}{\partial \bar{z}} = \delta(z, \bar{z}) \quad (7.27)$$

admits the subalgebra of  $L_\infty$  spanned by Operator 7.26 provided that

$$\alpha(0) = 0, \quad \beta(0, 0) = 0, \quad \eta(0) = \frac{\partial \alpha}{\partial z}(0). \quad (7.28)$$

The fundamental solution

$$u = \frac{1}{\pi z} \quad (7.29)$$

is an invariant solution of Equation 7.27 with respect to the subalgebra of  $L_\infty$  spanned by Operator 7.26 with Restraints 7.28

## 7.6. WAVE EQUATION IN RIEMANNIAN SPACE WITH A NONTRIVIAL CONFORMAL GROUP

$$u_{00} - u_{11} - \sum_{i,j=2}^n a^{ij} (x^1 - x^0) u_{ij} = 0. \quad (7.30)$$

Here  $(x^1, \dots, x^n)$  are spatial variables ( $n \geq 3$ ),  $x^0 = t$ ,  $u_{ij} = \partial^2 u / \partial x^i \partial x^j$  and  $\|a^{ij}\| = \|a_{ij}\|^{-1}$ .

The symmetry algebra  $L_r$ ,  $r = 2n + 1$ , is spanned by (Petrov [1961])

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^1}, & X_i &= \frac{\partial}{\partial x^i}, \\ Y_i &= -x^i \left( \frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^1} \right) + \sum_{j=2}^n A^{ij}(x^1 - x^0) \frac{\partial}{\partial x^j}, \\ Z_1 &= (x^0 + x^1) \left( \frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^1} \right) + \sum_{j=2}^n x^j \frac{\partial}{\partial x^j}, \\ Z_2 &= u \frac{\partial}{\partial u}, & (i, j &= 2, \dots, n), \end{aligned} \quad (7.31)$$

where  $\|A^{ij}\|$  is a matrix of functions  $A^{ij}(x^1 - x^0)$  such that

$$A^{ij}(\sigma) = \int a^{ij}(\sigma) d\sigma.$$

There exist coefficients  $a^{ij}(x^1 - x^0)$  of the special form when the symmetry algebra of Equation 7.45 admits an expansion.

The equation for fundamental solutions,

$$u_{00} - u_{11} - \sum_{i,j=2}^n a^{ij}(x^1 - x^0) u_{ij} = \delta(x - x_0), \quad (7.32)$$

where  $x_0 = (x_0^0, \dots, x_0^n)$ , admits the subalgebra of  $L_{2n+1}$  spanned by

$$\begin{aligned} Y_i &= -(x^i - x_0^i) \left( \frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^1} \right) + \sum_{j=2}^n [A^{ij}(x^1 - x^0) - A^{ij}(x_0^1 - x_0^0)] \frac{\partial}{\partial x^j}, \\ Z &= (x^0 + x^1 - x_0^0 - x_0^1) \left( \frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^1} \right) \\ &\quad + \sum_{j=2}^n (x^j - x_0^j) \frac{\partial}{\partial x^j} - (n-1)u \frac{\partial}{\partial u}. \end{aligned} \quad (7.33)$$

The fundamental solution

$$u = \frac{1}{2} \pi^{(1-n)/2} \left( \frac{|(x^1 - x_0^1) - (x^0 - x_0^0)|^{n-1}}{|\det \|A^{ij}(x^1 - x^0) - A^{ij}(x_0^1 - x_0^0)\|}|} \right)^{1/2} \delta^{(n-3)/2}(\Gamma),$$

where  $x_0 = (x_0^I, \dots, x_0^{\parallel})$ , admits the subalgebra of  $L$ , spanned by

$$\begin{aligned} \bar{X}_i &= \left[ (t^2 - |x|^2) - (t_0^2 - |x_0|^2) \right] \frac{\partial}{\partial x^i} + 2(x^i - x_0^i) \\ &\quad \times \left( t \frac{\partial}{\partial t} + \sum_1^n x^j \frac{\partial}{\partial x^j} \right) - (n-1)(x^i - x_0^i) u \frac{\partial}{\partial u}, \end{aligned} \quad (7.37)$$

$$\bar{X}_{ij} = (x^i - x_0^i) \frac{\partial}{\partial x^j} - (x^j - x_0^j) \frac{\partial}{\partial x^i} \quad (i < j).$$

The fundamental solution is given by

$$u = \frac{\theta(t - t_0)}{2\pi^{(n-1)/2}} F\left(-p, p(p+1), \frac{n+1}{2}, 1 + \frac{\Gamma}{4tt_0}\right) \delta^{(n-3)/2}(\Gamma), \quad (7.38)$$

where  $p \leq (n-3)/2$ ,  $\Gamma = (t - t_0)^2 - |x - x_0|^2$ , and  $F$  is Gauss' hypergeometric series:

$$\begin{aligned} F(\alpha, \beta, \gamma, x) \\ = 1 + \sum_{k=1}^{\infty} \frac{\alpha(\alpha+1) \cdots (\alpha+k-1) \beta(\beta+1) \cdots (\beta+k-1)}{\gamma(\gamma+1) \cdots (\gamma+k-1)} \frac{x^k}{k!}. \end{aligned}$$

This is a weak invariant solution of Equation 7.36 with respect to the group generated by Operators 7.37.

## 7.8. KLEIN-GORDON EQUATION

$$u_{tt} - \Delta_3 u + m^2 u = 0. \quad (7.39)$$

The symmetry algebra is spanned by

$$\begin{aligned} X_0 &= \frac{\partial}{\partial t}, & X_i &= \frac{\partial}{\partial x^i}, & X_{ij} &= x^j \frac{\partial}{\partial x^i} - x^i \frac{\partial}{\partial x^j}, \\ X_{0i} &= t \frac{\partial}{\partial x^i} + x^i \frac{\partial}{\partial t}, & Z &= u \frac{\partial}{\partial u}, & i &= 1, 2, 3. \end{aligned} \quad (7.40)$$

The fundamental solution

$$u = \frac{\Theta(t)}{2\pi} \delta(t^2 - |x|^2) - \frac{m}{4\pi} \Theta(t - |x|) \frac{J_1(m\sqrt{t^2 - |x|^2})}{\sqrt{t^2 - |x|^2}},$$

where  $J_1(x)$  is the Bessel function (see, e.g., Olver [1974]), is a weak invariant solution under the group generated by the operators  $X_{ij}, X_{0i}$ .

## 7.9. HELMHOLTZ EQUATION

$$\Delta_n u + k^2 u = 0, \quad (7.41)$$

where  $\Delta_n$  is the Laplacian in  $\mathbb{R}^n$ .

The symmetry algebra is spanned by

$$Z = u \frac{\partial}{\partial u}, \quad X_i = \frac{\partial}{\partial x^i}, \quad X_{ij} = x^j \frac{\partial}{\partial x^i} - x^i \frac{\partial}{\partial x^j}. \quad (7.42)$$

The fundamental solutions are given by

$$u = \frac{1}{2ik} e^{ik|x|}, \quad \bar{u} = -\frac{1}{2ik} e^{-ik|x|} \quad \text{for } n = 1,$$

$$u = -\frac{1}{4} H_0^{(1)}(k|x|), \quad \bar{u} = \frac{1}{4} H_0^{(2)}(k|x|) \quad \text{for } n = 2,$$

$$u = -\frac{e^{ik|x|}}{4\pi|x|}, \quad \bar{u} = -\frac{e^{-ik|x|}}{4\pi|x|} \quad \text{for } n = 3,$$

where  $H_0^{(\nu)}(x)$  is the Hankel function (see, e.g., Olver [1974]). They are invariant solutions under the group generated by  $X_{ij}$ .

## 7.10. HUYGENS' TYPE EQUATIONS RELATED TO COXETER GROUPS

(Berest and Veselov [1993a], [1993b], see also this volume, Chapter 4.)

The following hyperbolic operator is defined in  $\mathbb{R}^{10}$  by rank two Coxeter group of  $A_2$  - type:

$$L = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{(\partial y^1)^2} - \dots - \frac{\partial^2}{(\partial y^8)^2} - \frac{2}{t^2} - \frac{16(t^2 - 3x^2)}{(t^2 + 3x^2)^2}.$$

The fundamental solution of this operator is as follows:

$$u = \delta^{(3)}(\Gamma) \left( 1 - \frac{1}{2} \frac{3(tt_0 - xx_0)^2 + (xt_0 - tx_0)^2}{tt_0(t^2 + 3x^2)(t_0^2 + 3x_0^2)} \Gamma \right. \\ \left. + \frac{tt_0 - xx_0}{tt_0(t^2 + 3x^2)(t_0^2 + 3x_0^2)} \Gamma^2 - \frac{1}{2} \frac{1}{tt_0(t^2 + 3x^2)(t_0^2 + 3x_0^2)} \Gamma^3 \right),$$

where  $\Gamma = (t - t_0)^2 - (x - x_0)^2 - \sum_{i=1}^8 (y^i - y_0^i)^2$ .

The operator related to the Coxeter group of  $B_2$ -type is defined in  $\mathbb{R}^{12}$ :

$$L = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{(\partial y^1)^2} - \dots - \frac{\partial^2}{(\partial y^{10})^2} - \frac{2}{t^2} + \frac{2}{x^2} - \frac{8(t^2 - x^2)}{(t^2 + x^2)^2}.$$

The corresponding fundamental solution is as follows:

$$u = \delta^{(4)}(\Gamma) \left( 1 - \frac{1}{8} \frac{(xx_0 - tt_0)((tt_0 - xx_0)^2 + (xt_0 - t_0x)^2)}{tt_0xx_0(x^2 + t^2)(x_0^2 + t_0^2)} \Gamma \right. \\ \left. - \frac{1}{48} \frac{(x^2 + t^2)(x_0^2 + t_0^2) + 4(tt_0 - xx_0)^2 - 4xx_0tt_0}{tt_0xx_0(x^2 + t^2)(x_0^2 + t_0^2)} \Gamma^2 \right. \\ \left. + \frac{1}{16} \frac{xx_0 - tt_0}{tt_0xx_0(x^2 + t^2)(x_0^2 + t_0^2)} \Gamma^3 \right. \\ \left. + \frac{1}{32} \frac{1}{tt_0xx_0(x^2 + t^2)(x_0^2 + t_0^2)} \Gamma^4 \right),$$

where  $\Gamma = (t - t_0)^2 - (x - x_0)^2 - \sum_{i=1}^{10} (y^i - y_0^i)^2$ .

## 7.11. GENERALIZED AXISYMMETRICAL LAPLACE EQUATION

$$\frac{\partial^2 u}{(\partial x^0)^2} + \frac{\lambda}{x^0} \frac{\partial u}{\partial x^0} + \sum_{k=1}^n \frac{\partial^2 u}{(\partial x^k)^2} = 0, \quad n \geq 2, \quad (7.43)$$

where  $\lambda$  is a real constant.

The symmetry algebra  $L_r$ ,  $r = (n + 1)(n + 2)/2 + 1$ , is spanned by (Aksenov [1993]):

$$X_i = \frac{\partial}{\partial x^i}, \quad X_{ij} = x^j \frac{\partial}{\partial x^i} - x^i \frac{\partial}{\partial x^j}, \\ Z_1 = x^j \frac{\partial}{\partial x^j}, \quad Z_2 = u \frac{\partial}{\partial u}, \quad (i, j = 1, \dots, n), \\ Y_i = 2x^0 x^i \frac{\partial}{\partial x^0} + \sum_{j=1}^n \left[ 2x^i x^j - \delta^{ij} \sum_{k=0}^n (x^k)^2 \right] \frac{\partial}{\partial x^j} \\ - (\lambda + n - 1) x^i u \frac{\partial}{\partial u}. \quad (7.44)$$

The equation for fundamental solutions,

$$\frac{\partial^2 u}{(\partial x^0)^2} + \frac{\lambda}{x^0} \frac{\partial u}{\partial x^0} + \sum_{k=1}^n \frac{\partial^2 u}{(\partial x^k)^2} = \delta(x^0 - 1)\delta(x), \quad n \geq 2, \quad (7.45)$$

where  $x = (x^1, \dots, x^n)$ , admits the subalgebra spanned by

$$\begin{aligned} X_{ij} &= x^j \frac{\partial}{\partial x^i} - x^i \frac{\partial}{\partial x^j} \quad (i < j, i = 1, \dots, n, j = 2, \dots, n), \\ X_i &= \left[ 1 - \sum_{k=0}^n (x^k)^2 \right] \frac{\partial}{\partial x^i} + 2x^i \sum_{k=0}^n x^k \frac{\partial}{\partial x^k} - (\lambda + n - 1)x^i u \frac{\partial}{\partial u}. \end{aligned} \quad (7.46)$$

The fundamental solution

$$u = \rho^{1-n} \rho_1^{-\lambda} \left( 1 - \frac{\rho^2}{\rho_1^2} \right)^{1-\lambda} F \left( 2 - \frac{n + \lambda + 1}{2}, 1 - \frac{\lambda}{2}, 2 - \lambda, 1 - \frac{\rho^2}{\rho_1^2} \right)$$

is a weak invariant solution of Equation 7.45 with respect to the group generated by Operators 7.46. Here

$$\rho^2 = (x^0 - 1)^2 + \sum_{i=1}^n (x^i)^2, \quad \rho_1^2 = (x^0 + 1)^2 + \sum_{i=1}^n (x^i)^2$$

and  $F$  is Gauss' hypergeometric function considered in Section 7.7.

# Classical Field Theory

## 8.1. NOTATION

We used throughout this Chapter the special units  $\hbar = e = c = 1$ , where  $\hbar$  is the Planck constant,  $e$  is the electron charge, and  $c$  is the light velocity. We also apply the conventional quantum mechanical notation for the *momentum operators*:

$$p_k = i \frac{\partial}{\partial x^k}, \quad P_k = p_k + A_k,$$

where  $i^2 = -1$ , and  $A_k$  is a *vector potential* of an electromagnetic field. The scalar product in the Minkowsky space is denoted by  $(xx) = x^k x_k = x_0^2 - r^2$ ,  $(xp) = x^k p_k$ , etc. Here  $x_i = \eta_{ij} x^j$ ,  $\mathbf{r}$  is the position vector,  $\eta_{ij} = \text{diag}\{1, -1, -1, -1\}$  is the Minkowski metric tensor,  $\eta_{ij} \eta^{jk} = \delta_i^k$ , and  $\delta_i^k$  is the *Kronecker symbol*. The summation rule over the repeated indices is used throughout. Let  $L$  be a matrix linear differential operator,  $u = (u^1, \dots, u^m)^T$  be the column of the dependent variables where  $T$  denotes the transposition. If the linear system

$$L(x, p)u = 0, \tag{8.1}$$

admits the *canonical Lie-Bäcklund operator* (see Formula 5.22 from [H1]), then the *determining equation* takes the form

$$L\eta(u) = B(u)Lu, \tag{8.2}$$

where  $B$  is a linear partial differential operator.

If Equation 8.1 is the Klein-Gordon system or the Laplace-Beltrami equation, then  $\eta(u) = Xu$ , where  $X$  is a linear differential operator. (Meshkov [1983a], Shapovalov and Shirokov [1992]).

Usually the assumption  $\eta(u) = Xu$  is adopted for any linear system. Then Equation 8.2 is reduced to the following operator on (Shapovalov [1968a]):

$$LX = BL. \quad (8.3)$$

Operator  $X$  satisfying Equation 8.3 is called the *symmetry operator*. There is the following important property: any polynomial of the symmetry operators is a new symmetry operator. The set of the symmetry operators is wider than the set of the Lie generators. For instance, several integral symmetry operators were obtained by Fushchich and Nikitin [1983] and Fushchich, Shtelen, and Serov [1989]. Any linear equation possesses the obvious symmetry operator  $X = 1$ , that corresponds to the scale symmetry  $u' = u \exp(a)$ . Besides, any linear equation 8.1 admits the following symmetry transformation  $u' = u + f(x)$ , where  $f$  solves Equation 8.1.

## 8.2. WAVE EQUATION

$$\square u \equiv \frac{\partial^2 u}{\partial (x^0)^2} - \frac{\partial^2 u}{\partial (x^1)^2} - \frac{\partial^2 u}{\partial (x^2)^2} - \frac{\partial^2 u}{\partial (x^3)^2} = 0. \quad (8.4)$$

The symmetry operators are given by (cf. Section 7.3):

$$p_k, \quad L_{ij} = x_i p_j - x_j p_i, \quad (8.5)$$

$$D = (xp), \quad K_j = 2x_j(xp) - (xx)p_j + 2ix_j. \quad (8.6)$$

Operators 8.5 generate the *Poincaré group*,  $D$  generates the dilatations and the operators  $K_i$  generate the special conformal transformations

$$x'^i = \frac{x^i + c^i(xx)}{1 + 2(cx) + (cc)(xx)}, \quad u' = \frac{u}{1 + 2(cx) + (cc)(xx)}, \quad (8.7)$$

where  $c$  is a constant vector. Lorentz invariance of the wave equation was established by Poincaré [1906]. Transformation 8.7 was established by

Bateman [1909] in the following form. If  $(cc) \neq 0$ , then Transformation 8.7 is a superposition of the Poincaré transformations, dilatations and the inversion:

$$x'^i = x^i / (xx), \quad u' = u / (xx). \quad (8.8)$$

If  $(cc) = 0$ , then Transformation 8.7 is a superposition of the Poincaré transformations, dilatations and the following transformation:

$$x'^0 = \frac{(xx) - 1}{2(x^0 - x^1)}, \quad x'^1 = \frac{(xx) + 1}{2(x^0 - x^1)}, \quad (8.9)$$

$$x'^i = \frac{x^i}{x^0 - x^1}, \quad i = 2, 3, \quad u' = \frac{u}{x^0 - x^1}.$$

Notice that this structure of the conformal group was known for Lie [1871], page 351.

### 8.2.1. NONEQUIVALENT OPERATORS

There exist nineteen first-order symmetry operators that are nonequivalent under the conformal group (Bagrov et al [1988]):

1.  $L_{23} + \alpha p_1$ ;    2.  $L_{23} + \alpha p_0$ ;    3.  $L_{23} + \alpha(p_1 + p_0)$ ;    4.  $p_1$ ;
5.  $p_0 + p_1$ ;    6.  $p_0$ ;    7.  $L_{03} + \alpha p_1$ ;    8.  $L_{23} + \alpha L_{01}$ ,  $\alpha \neq 0$ ;
9.  $L_{23} + L_{02} + \alpha p_1$ ;    10.  $L_{23} + L_{02} + \alpha(p_0 + p_3)$ ,  $\alpha \neq 0$ ;
11.  $K_0 + p_1 + \alpha(K_1 - p_0) + \beta L_{23}$ ;    12.  $K_0 + p_0 + \alpha(K_1 - p_1)$ ;
13.  $K_0 + K_1 + p_0 - p_1 + 2L_{32} + \alpha(p_2 + L_{03} + L_{13})$ ;
14.  $K_0 + K_1 + p_0 - p_1 + \alpha(p_0 + p_1) + \beta L_{23}$ ;
15.  $D - L_{01} + \alpha L_{23} + \beta(p_0 + p_1)$ ;    16.  $D + \alpha L_{01} + \beta L_{23}$ ;
17.  $K_1 - p_1 + \alpha L_{23}$ ;    18.  $K_1 - p_0 + \alpha L_{23}$ ;    19.  $\sqrt{2}K_2 + p_0 + p_3$ .

### 8.2.2. IDENTITIES AND HIGHER SYMMETRIES

The complete list of the quadratic identities was given by Bagrov et al. [1990]:

$$\begin{aligned}
 L_{ij}p_k + L_{jk}p_i + L_{ki}p_j &= 0, & 2L_{ij}D + K_jp_i - K_ip_j &= 0, \\
 \epsilon_{ijkl}L^{ij}L^{kl} &= 0, & L_{ij}K_k + L_{jk}K_i + L_{ki}K_j &= 0, & i \neq j \neq k, \\
 (1/2)L_{ij}L^{ij} + D^2 + 2iD &= -(xx)\square, \\
 \eta^{kl}L_{jk}L_{il} + K_jp_i + i\eta_{ij}D &= -x_j^2\square, & (KK) &= -(xx)^2\square, \\
 2\eta^{kl}\{L_{ik} \cdot L_{jl}\} + \{K_i \cdot p_j\} + \{K_j \cdot p_i\} &= -x_ix_j\square, & i \neq j, \\
 L_{jk}p^k + Dp_j &= -x_j\square, & K^iL_{ij} + K_jD &= -(1/2)x_j(xx)\square,
 \end{aligned} \tag{8.11}$$

where  $\epsilon_{ijkl}$  is the *Levi-Civita symbol*, ( $\epsilon_{0123} = 1$ ), and  $\{a \cdot b\} = (1/2)(ab + ba)$  is the *Jordan product*.

Any higher symmetry operator for the wave equations is a polynomial of the operators 8.5 and 8.6 plus *trivial operator*:  $A = R(x, p)\square$  (Bagrov et al. [1990], Nikitin [1991a]).

### 8.2.3. WAVE EQUATION IN $n$ DIMENSIONS

For any constant metric  $\eta_{ij}$  in  $n$  dimensions the symmetry operators  $p_i$ ,  $L_{ij}$ , and  $D$  are the same but  $K_j$  take the following form

$$K_j = 2x_j(xp) - (xx)p_j + i(n-2)x_j. \tag{8.12}$$

This implies the following modification in the transformation formula 8.7 for  $u$ :

$$u' = u\{1 + 2(cx) + (cc)(xx)\}^{(2-n)/2}. \tag{8.13}$$

For the metric  $\eta_{ij} = \text{diag}\{1, -1, \dots, -1\}$  Formulas 8.8 and 8.9 are modified as follows (Carmichael [1927]):

$$x'^i = x^i/(xx), \quad u' = u(xx)^{(2-n)/2}, \tag{8.14}$$

and

$$x'^0 = \frac{(xx) - 1}{2(x^0 - x^1)}, \quad x'^1 = \frac{(xx) + 1}{2(x^0 - x^1)}, \quad x'^i = \frac{x^i}{x^0 - x^1}, \quad (8.15)$$

$$i = 2, \dots, n-1, \quad u' = u(x^0 - x^1)^{(2-n)/2}.$$

#### 8.2.4. CONSERVATION LAWS

Any conserved current for the wave equation takes the following form:

$$I_k = u_k Xu - u \frac{\partial}{\partial x^k} Xu, \quad (8.16)$$

where  $X$  is a symmetry operator for Equation 8.4 (Abellanas and Galindo [1981]).

Choosing  $X = p_i$  or  $L_{ij}$  here, we obtain the following conserved currents:

$$\begin{aligned} \tilde{T}_i^k &= u^k u_i - u \eta^{kj} u_{ij}, \quad u^k = \eta^{ki} u_i, \\ \tilde{M}_{ij}^k &= x_i \tilde{T}_j^k - x_j \tilde{T}_i^k + u(\delta_i^k u_j - \delta_j^k u_i). \end{aligned} \quad (8.17)$$

Tensors  $\tilde{T}_{ik}$  and  $\tilde{M}_{ij}^k$  are equivalent to the conventional tensors of the energy-momentum and angular momentum  $T_{ik}$  and  $M_{ij}^k$ , respectively:

$$\begin{aligned} \tilde{T}_i^k &= T_i^k + \frac{\partial}{\partial x^j} A_i^{kj}, \quad \tilde{M}_{ij}^k = -M_{ij}^k + \frac{\partial}{\partial x^l} B_{ij}^{kl}, \\ A_i^{jk} &= -A_i^{kj} = u(\delta_i^j u^k - \delta_i^k u^j), \\ B_{ij}^{kl} &= -B_{ij}^{lk} = x_i A_j^{kl} - x_j A_i^{kl} + 2u^2(\delta_i^k \delta_j^l - \delta_j^k \delta_i^l), \end{aligned} \quad (8.18)$$

where  $T_{ik}$  and  $M_{ij}^k$  are given by (cf. Expressions 8.32)

$$T_i^k = 2u^k u_i - \delta_i^k u_s u^s, \quad M_{ij}^k = T_i^k x_j - T_j^k x_i. \quad (8.19)$$

Operators 8.6  $D$  and  $K_j$  imply the additional conserved currents

$$B^i = x^s T_s^i + 2uu^i, \quad A_j^i = 2x_j B^i - (xx)T_j^i - 2u^2 \delta_j^i, \quad (8.20)$$

respectively. Any higher conserved current take form 8.16, where  $X$  is a polynomial of Operators 8.5 and 8.6 as was mentioned previously.

### 8.3. KLEIN-GORDON EQUATION

$$[\eta^{ij}P_iP_j - m^2]u = 0. \quad (8.21)$$

Notation are given in Section 8.1;  $m > 0$  is a parameter. The equivalence group for Equation 8.21 is the direct product of the Poincaré group and the Abelian gauge group:

$$\begin{aligned} x'^i &= a_j^i x^j + c^i, & a_j^i a_l^k \eta_{ik} &= \eta_{jl}, \\ u' &= u \exp(iS), & A'_k &= A_k + (\partial S / \partial x^k), \end{aligned} \quad (8.22)$$

where  $S$  is an arbitrary function.

#### 8.3.1. FREE EQUATION

If  $A_k = 0$ , then Equation 8.21 possesses only ten symmetry operators of the first-order 8.5 (Ovsiannikov [1962]). That operators generate the Poincaré group as it was mentioned in Section 8.2. Any *higher symmetry operator* of the free equation 8.21 is a polynomial of Operators 8.5 plus *trivial operator*:  $A = R(x, \partial/\partial x)(\square + m^2)$ , (Bagrov et al. [1990]; Nikitin and Prilipko [1990] and Nikitin [1991a]).

There exist ten independent, under the Poincaré group, symmetry operators (cf. Expressions 8.18) (Bagrov et al. [1973]; Patera, Winternitz and Zassenhaus [1975]):

$$\begin{aligned} 1. & L_{23} + \alpha p_1; & 2. & L_{23} + \alpha p_0; & 3. & L_{23} + \alpha(p_1 + p_0); & 4. & p_1; \\ 5. & p_0 + p_1; & 6. & p_0; & 7. & L_{03} + \alpha p_1; & 8. & L_{23} + \alpha L_{01}, \alpha \neq 0; \\ 9. & L_{23} + L_{02} + \alpha p_1; & 10. & L_{23} + L_{02} + \alpha(p_0 + p_3), \alpha \neq 0. \end{aligned} \quad (8.23)$$

#### 8.3.2. NONFREE EQUATION

Each symmetry operator of the first order is equivalent, under the group 8.22, to one of the operators 8.23 (Bagrov et al. [1973]). If  $X = X^i p_i$  is a symmetry operator for Equation 8.21, then the vector fields  $X^i$  and  $A^i$  are commutative. The *admissible potentials* for the operators 8.23 take the following form:

$$\begin{aligned} 1-3. & \quad A_0 = A_0(\xi), A_1 = A_1(\xi), A_2 = x^2 a(\xi) + x^3 b(\xi), \\ & \quad A_3 = x^3 a(\xi) - x^2 b(\xi), \text{ where} \end{aligned}$$

1.  $\xi_1 = x^0, \xi_2 = x_2^2 + x_3^2, \xi_3 = x^1 + \alpha \arctan(x^3/x^2);$
2.  $\xi_1 = x^1, \xi_2 = x_2^2 + x_3^2, \xi_3 = x^0 + \alpha \arctan(x^3/x^2);$
3.  $\xi_1 = x^0 - x^1, \xi_2 = x_2^2 + x_3^2, \xi_3 = x^0 + x^1 + 2\alpha \arctan(x^3/x^2);$
- 4-6.  $A_i = A_i(\xi)$ , where, 4.  $\xi_1 = x^0, \xi_2 = x^2, \xi_3 = x^3;$
5.  $\xi_1 = x^0 - x^1, \xi_2 = x^2, \xi_3 = x^3;$  6.  $\xi_i = x^i, i \neq 0;$
7.  $A_0 = (x^0 - x^3)a(\xi) + (x^0 + x^3)b(\xi), A_1 = A_1(\xi),$   
 $A_3 = (x^0 - x^3)(\xi) - (x^0 + x^3)b(\xi), A_2 = A_2(\xi),$  where  
 $\xi_1 = x^1 + \frac{\alpha}{2} \ln \left| \frac{x^0 - x^3}{x^0 + x^3} \right|, \xi_2 = x^2, \xi_3 = x_0^2 - x_3^2;$
8.  $A_0 = (x^0 - x^1)a(\xi) + (x^0 + x^1)b(\xi), A_2 = x^2f(\xi) + x^3g(\xi),$   
 $A_1 = (x^0 - x^1)a(\xi) - (x^0 + x^1)b(\xi), A_3 = x^3f(\xi) - x^2g(\xi),$   
 where  $\xi_1 = x_0^2 - x_1^2,$   
 $\xi_2 = x_2^2 + x_3^2, \xi_3 = \alpha \arctan(x^3/x^2) + \frac{1}{2} \ln \left| \frac{x^0 + x^1}{x^0 - x^1} \right|; \quad (8.24)$
9.  $A_0 = [x_2^2 + (x^0 + x^3)^2]a(\xi) - x^2b(\xi) + f(\xi),$   
 $A_1 = A_1(\xi), A_2 = (x^0 + x^3)[b(\xi) - 2x^2a(\xi)],$   
 $A_3 = [x_2^2 - (x^0 + x^3)^2]a(\xi) - x^2b(\xi) + f(\xi),$  where  
 $\xi_1 = x^1(x^0 + x^3) - \alpha x^2, \xi_2 = x_0^2 - x_2^2 - x_3^2, \xi_3 = x^0 + x^3;$
10.  $A_0 = [(x^0 + x^3)^2 + 4\alpha^2]a(\xi) - (x^0 + x^3)b(\xi) + f(\xi),$   
 $A_3 = [(x^0 + x^3)^2 - 4\alpha^2]a(\xi) - (x^0 + x^3)b(\xi) + f(\xi),$   
 $A_1 = A_1(\xi), A_2 = 2\alpha b(\xi) - 4\alpha(x^0 + x^3)a(\xi),$   
 where  $\xi_1 = x^1, \xi_2 = (x^0 + x^3)^2 - 4\alpha x^2,$   
 $\xi_3 = (x^0 + x^3)^3 + 6\alpha^2(x^0 - x^3) - 6\alpha x^2(x^0 + x^3).$

**Example.** Let  $X = p_0 + p_1$  be the symmetry operator and  $\lambda$  is a parameter. Then the equation  $Xu = \lambda u$  determines the *quantum states*

$$u = \psi(x^2, x^3, x^0 - x^1) \exp[-(i/2)\lambda(x^0 + x^1)]$$

with the definite values of the physical value  $p_0 + p_1$ . It is obvious that the equation  $Xu = \lambda u$  determines the invariant manifold for the operator

$$Y = i \frac{\partial}{\partial x^0} + i \frac{\partial}{\partial x^1} + \lambda u \frac{\partial}{\partial u}.$$

In the coordinates

$$x'^3 = (1/2)(x^0 + x^1), \quad x'^0 = (1/2)(x^0 - x^1), \quad x'^1 = x^3, \quad x'^2 = x^2,$$

Equation 8.21 is reduced to the following equation for  $\psi$ :

$$(ap'_0 - P_1'^2 - P_2'^2 + V)\psi = 0, \quad (8.25)$$

where the functions  $a = A_0 + A_1 + \lambda$ ,  $V = (A_0 - A_1)a + iF_{01} - m^2$ , and  $F_{01} = (\partial A_1 / \partial x^0) - (\partial A_0 / \partial x^1)$  do not depend on  $x'^3$  according to the expression 5 in Formulas 8.24. It is remarkable that Equation 8.25 possesses the infinite equivalence group (Shapovalov [1974]):

$$\tilde{x}^0 = f(x'^0, \lambda), \quad \tilde{x}^i = a_j^i(x'^0, \lambda)x'^j + b^i(x'^0, \lambda), \quad \tilde{\lambda} = \phi(\lambda),$$

where  $a_j^i$  is an orthogonal matrix.

### 8.3.3. CONSERVATION LAWS

Any conserved current for Equation 8.21 takes the following form:

$$I^k = (P^k u)^* Xu + u^* P^k Xu, \quad (8.26)$$

where symbol  $*$  denotes the complex conjugation, and  $X$  is a symmetry operator for Equation 8.21 (Meshkov [1993a]; for the free equation see Abellanas and Galindo [1981] and Galindo [1981]).

This result admits the generalization onto the Klein-Gordon system

$$P^i P_i u^\alpha = M_{\alpha\beta} u^\beta, \quad (8.27)$$

where  $M$  is a symmetric constant matrix.

Any conserved current for the system 8.27 takes the following form:

$$I^k = (P^k u)^+ Xu + u^+ P^k Xu, \quad (8.28)$$

where  $u^+ = (u^*)^T$  denotes the Hermitian conjugation, and  $X$  is a symmetry operator for the system 8.27.

Currents 8.26 and 8.28 are complex, therefore  $I_k \pm I_k^*$  are the conserved currents too. The free equation has real solutions therefore the conserved currents read

$$I_k = u_k Xu - u \frac{\partial}{\partial x^k} Xu. \quad (8.29)$$

Choosing  $X = 1$ ,  $p_i$  and  $L_{ij}$  in 8.27, we obtain the following conserved currents for the free system:

$$J_k = i(u^+ u_k - u_k^+ u), \quad \tilde{T}_i^k = \eta^{kj}(u_j^+ u_i - u^+ u_{ij}), \quad (8.30)$$

$$\tilde{M}_{ij}^k = x_i \tilde{T}_j^k - x_j \tilde{T}_i^k + u^+ (\delta_i^k u_j - \delta_j^k u_i).$$

Here  $J_k$  is the *electric current*. Tensors  $\tilde{T}_{ik}$  and  $\tilde{M}_{ij}^k$  can be represented in the following form:

$$\text{Re } \tilde{T}_i^k = T_i^k + \partial_j A_i^{kj}, \quad \text{Re } \tilde{M}_{ij}^k = -M_{ij}^k + \partial_l B_{ij}^{kl},$$

$$A_i^{jk} = -A_i^{kj} = (1/2)(\delta_i^j \partial^k - \delta_i^k \partial^j) u^+ u, \quad \partial^j = \eta^{jk} \partial / \partial x^k, \quad (8.31)$$

$$B_{ij}^{kl} = -B_{ij}^{lk} = x_i A_j^{kl} - x_j A_i^{kl} + 2(\delta_i^k \delta_j^l - \delta_j^k \delta_i^l) u^+ u.$$

Here  $T_{ik}$  and  $M_{ij}^k$  are the conventional tensors of the energy-momentum and angular momentum:

$$T_i^k = (u^+)^k u_i + u_i^+ u^k - \delta_i^k L, \quad M_{ij}^k = T_i^k x_j - T_j^k x_i, \quad (8.32)$$

where  $L = u_i^+ u^i - u^+ Mu$  is the Lagrangian. Equations 8.31 denote that  $\tilde{T}_{ik}$  is equivalent to  $T_{ik}$  and  $\tilde{M}_{ij}^k$  is equivalent to  $M_{ij}^k$ . Any higher conserved current for the system 8.27 takes the form 8.28, where  $X$  is a polynomial of the operators 8.5 as it was mentioned previously. Several explicit expressions for the higher conserved currents of the free equation 8.21 were presented by Gordon [1981].

## 8.4. NONSTATIONARY SCHRÖDINGER EQUATION

$$Hu \equiv \left[ i \frac{\partial}{\partial t} - \frac{1}{2m} \sum_{k=1}^n P_k^2 - V \right] u = 0. \quad (8.33)$$

Here  $n = 3$  for one particle and  $n = 3N$  for  $N$ -body problem.

### 8.4.1. FREE EQUATION

If  $A_k = V = 0$  then Equation 8.33 possesses the following first-order symmetry operators (Niederer [1972]):

$$\begin{aligned} p_0 &= i \frac{\partial}{\partial t}, & p_j &= i \frac{\partial}{\partial x^j}, & A &= t^2 p_0 - tD - (m/2)r^2 - (ni/2)t, \\ L_{jk} &= x_j p_k - x_k p_j, & G_k &= t p_k + m x^k, & D &= 2t p_0 + x^k p_k. \end{aligned} \quad (8.34)$$

Here  $j, k = 1, \dots, n$ ,  $x_k = x^k$ ,  $r^2 = x_k x^k$ . The symmetry operators  $p_k$ ,  $k \geq 0$ , and  $L_{jk}$  are the generators of the translations and rotations respectively.  $D$  is the generator of the dilatations  $x'^i = x^i \exp(a)$ ,  $t' = t \exp(2a)$ . Operators  $G_k$  correspond to the generators of the *Galilei group*:

$$\hat{G}_k = t \frac{\partial}{\partial x^k} + i m x^k u \frac{\partial}{\partial u}. \quad (8.35)$$

The Galilean transformations are given by

$$t' = t, \quad x'^k = x^k + t v^k, \quad u' = u \exp\{im(\mathbf{r}\mathbf{v} + t \mathbf{v}^2/2)\}, \quad (8.36)$$

where  $\mathbf{v}$  is the vector of the group parameters. Operator  $A$  corresponds to the Lie generator

$$\hat{A} = t^2 \frac{\partial}{\partial t} + t x^k \frac{\partial}{\partial x^k} + (1/2)(imr^2 - nt)u \frac{\partial}{\partial u}. \quad (8.37)$$

The corresponding group is given by

$$t' = \frac{t}{1 - at}, \quad x'^i = \frac{x^i}{1 - at}, \quad u' = u(1 - at)^{n/2} \exp\left[\frac{imar^2}{2(1 - at)}\right], \quad (8.38)$$

where  $a$  is the group parameter.

Any higher symmetry operator of the free equation is a polynomial of the operators 8.34 (Nikitin [1991b]). Operators  $A$ ,  $G_k$ , and  $D$  do not commute with  $H$  but if we substitute  $p_0 = \mathbf{p}^2/(2m)$  into Expressions 8.34, then the transformed operators 8.34 will commute with  $H$ .

### 8.4.2. NONFREE EQUATION

Equivalence group for Equation 8.33 is given by

$$t' = f(t), \quad x'^i = a_j^i(t)x^j + b^i(t), \quad u' = u \exp(iS), \quad (8.39)$$

where  $f$ ,  $b^i$ , and  $S$  are arbitrary functions;  $a_j^i = aO_j^i$ ,  $O_j^i$  is an orthogonal matrix and  $a^2 = \dot{f} \equiv df/dt$ . The transformation formulas for the potentials take the following form:

$$\begin{aligned} A'^i &= a^{-2} \left[ a_s^i A^s - m(\partial x'^i / \partial t) \right] - (\partial S / \partial x'^i), \\ V' &= (\partial S / \partial t') + a^{-2} V \\ &\quad + a^{-4} \left[ a_s^k A^s (\partial x'^k / \partial t) - \frac{m}{2} \sum_k (\partial x'^k / \partial t)^2 + \frac{i}{2} n \dot{a} a \right]. \end{aligned} \quad (8.40)$$

Formulas 8.39 and 8.40 slightly generalize that given by Shapovalov [1974]. Tensor  $F_{ij} = (\partial A_j / \partial x^i) - (\partial A_i / \partial x^j)$ , where  $i, j \geq 0$ ,  $A_0 = -V$ ,  $x^0 \equiv t$ , is called the *strength tensor* of the electromagnetic field. It is the gauge invariant value. Formulas 8.40 imply that the field

$$\begin{aligned} F_{i0} &= \frac{1}{2} \left( \dot{F}_{is} - \frac{1}{2m} F_{ij} F_{js} \right) x^s + m \dot{a}^2(t) a^{-2} x_i + f_i(t), \\ F_{ij} &= F_{ij}(t), \quad i, j > 0, \end{aligned} \quad (8.41)$$

is equivalent to zero field  $F'_{ij} = F'_{i0} = 0$ . To obtain the coordinates, where the field is zero, one ought to solve the following system:

$$\dot{O}_k^i(t) = \frac{1}{2m} F_{ks} O_s^i, \quad \frac{d}{dt} (a^{-2} \dot{b}^i(t)) = -\frac{1}{m} a^{-2} a_k^i f^k, \quad (8.42)$$

where  $OO^T = E$  and  $a_j^i = aO_j^i$ . It is clear now that Equation 8.33 with the field 8.41 possesses the symmetry algebra isomorphic to 8.34. If the potentials

are given by

$$\begin{aligned} A_k &= (1/2)x^j F_{jk}(t) + m\dot{a}a^{-1}x_k + ma^{-2}a_k^i \dot{b}_i, \\ 2V &= (1/(4m))F_{is}F_{sj}x^i x^j + a^{-2}a_j^i \dot{b}_i F_{js}x^s - m\dot{a}^2 a^{-2}r^2 \\ &\quad - 2m\dot{a}a^{-3}\dot{b}_i a_j^i x^j - ma^{-2}\dot{\mathbf{b}}^2 - in\dot{a}a^{-1}, \end{aligned} \quad (8.43)$$

where  $a_j^i$  and  $b^i$  are the solution of System 8.42, then the symmetry operators for Equation 8.33 are:

$$\begin{aligned} \tilde{p}_0 &= a^{-2}p_0 - a^{-4}(\dot{a}_s^i x^s + \dot{b}^i)a_k^i p_k, \\ \tilde{D} &= (1 - 2\dot{a}a^{-3}f)x^i p_i + 2a^{-2}fp_0 \\ &\quad + a^{-2}[a_k^i(b_i - 2fa^{-2}\dot{b}_i) - (1/m)fx^j F_{jk}]p^k, \\ \tilde{A} &= f^2 a^{-2}p_0 - f^2 a^{-4}(\dot{a}_s^i x^s + \dot{b}^i)a_k^i p^k - f\tilde{D} - (in/2)f \\ &\quad - (1/2)m(a^2 r^2 + \mathbf{b}^2 + 2a_j^i b_i x^j), \\ \tilde{G}_k &= fa^{-2}a_s^k p^s + m(a_s^k x^s + b^k), \\ \tilde{p}_i &= a^{-2}a_k^i p^k, \quad \tilde{L}_{ij} = a^{-2}a_k^i a_l^j L_{kl} + a^{-2}(b^i a_k^j - b^j a_k^i)p^k. \end{aligned} \quad (8.44)$$

Any polynomial of these operators is a symmetry operator. Any first-order symmetry operator takes the following form ( $n = 3$ ):

$$X = 2\chi p_0 + \dot{\chi}(\mathbf{r}\mathbf{p}) + (\mathbf{a}\mathbf{L}) + (\mathbf{b}\mathbf{p}),$$

where  $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ , scalar  $\chi$  and the vectors  $\mathbf{a}, \mathbf{b}$  depend on  $t$  only. Nonequivalent under Group 8.39 operators and the corresponding admissible potentials are given by Shapovalov [1974]:

1.  $X = p_0, \quad \partial A_k / \partial t = \partial V / \partial t = 0;$
2.  $X = \alpha(t)p_3, \quad A_1 = 0, \quad A_2 = a_2, \quad A_3 = \alpha^{-1}(a_3 - m\dot{\alpha}x_3);$   
 $V = (x^3/\alpha)(\partial A/\partial t) - m\ddot{\alpha}(x_3^2/(2\alpha)) + a_0, \quad a_i = a_i(t, x^1, x^2)$

(iv)  $\gamma < 0$ ,  $\beta = \sinh^{-1} \lambda t \frac{d}{dt}(\dot{b} \sinh^2 \lambda t)$ , where  $b$  is an arbitrary function and  $\lambda = \sqrt{-2\gamma/m}$ .

$$x' = x \sinh^{-1} \lambda t + b/m, \quad t' = -(1/\lambda) \coth \lambda t,$$

$$S = -\frac{i}{2} \ln \sinh \lambda t + \frac{1}{2m} \int \dot{b}^2 \sinh^2 \lambda t dt - \frac{m\lambda}{2} x^2 \coth \lambda t + \dot{b}x \sinh \lambda t. \quad (8.55)$$

3.  $V = \beta(t)x$ . Then there exist two transformations to the free equation.  
(i)  $\beta = \ddot{b}(t)$ , where  $b$  is an arbitrary function.

$$x' = x + b/m, \quad t' = t, \quad S = \frac{1}{2m} \int \dot{b}^2 dt + \dot{b}x. \quad (8.56)$$

(ii)  $\beta = t^{-1} \frac{d}{dt}(\dot{b}t^2)$ , where  $b$  is an arbitrary function.

$$x' = \frac{x}{t} + \frac{b}{m}, \quad t' = -\frac{1}{t}, \quad (8.57)$$

$$S = -\frac{i}{2} \ln t + \frac{1}{2m} \int \dot{b}^2 t^2 dt - \frac{m}{2t} x^2 + \dot{b}xt.$$

Equation 8.48 admits Operators 8.51 for each of the cases 8.52–8.57. Any higher symmetry operator is a polynomial of the operators 8.51.

4.  $V = ax^{-2} + bx^2$ , where  $a$  and  $b$  are any constants,  $a \neq 0$ . There are only three independent symmetry operators of the first order

$$X_1 = p_0, \quad X_2 = 2tp_0 + xp_1 + i/2, \quad (8.58)$$

$$X_3 = t^2 p_0 - tX_2 - mx^2/2.$$

Each higher symmetry operator is a polynomial of the operators 8.58 (Nikitin, Onufrychuk, and Fushchich [1992]).

Other examples see in Anderson, Kumei, and Wulfman [1972], Bagrov and Gitman [1982], Beckers, Debergh, and Nikitin [1991], and in the references cited therein.

## 8.5. STATIONARY SCHRÖDINGER EQUATION

$$H\psi \equiv \left[ \frac{1}{2} \sum_{i=1}^n P_i^2 + V \right] \psi = E\psi. \quad (8.59)$$

Notation are given in the Section 8.1. Besides  $V$  is a scalar potential and  $E$  is the eigenvalue of the operator  $H$ . The equivalence group of Equation 8.59 is

the direct product of the *Euclidean group* and the Abelian gauge group:

$$x'^i = a_j^i x^j + c^i, \quad a_j^i a_l^k \delta_{ik} = \delta_{jl}, \quad (8.60)$$

$$u' = u \exp(iS), \quad A'_k = A_k + (\partial S / \partial x^k),$$

where  $S$  is an arbitrary function. The first-order symmetry operators for the free equation are  $p_i$  and  $L_{ij}$ . They generate the translations and rotations. If  $n = 3$ , then each first-order symmetry operator for the nonfree equation 8.59 is equivalent under the group 8.60 to one of the following operators:

$$1. \quad p_3; \quad 2. \quad L_3 + \alpha p_3, \quad (8.61)$$

where  $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ ,  $\alpha$  is a constant (Bagrov et al. [1972]). The correspondent admissible potentials are given by

$$1. \quad A_i = A_i(x^1, x^2), \quad V = V(x^1, x^2),$$

$$2. \quad A_1 = a_1 \cos \varphi - a_2 \sin \varphi, \quad A_2 = a_1 \sin \varphi + a_2 \cos \varphi, \quad (8.62)$$

$$A_3 = 0, \quad a_i = a_i(\rho, \xi), \quad V = V(\rho, \xi),$$

where  $\rho = \sqrt{x_1^2 + x_2^2}$ ,  $\varphi = \arctan(x^2/x^1)$ ,  $\xi = x^3 - \alpha\varphi$ .

For the bound states the wave function  $\psi$  and the eigenvalue  $E$  depend on several integers  $n_1, n_2, \dots$  that are called the *quantum numbers*. If two or more solutions  $\psi(n_i)$  exist for the same eigenvalue  $E_0 = E(n_i)$ , then they say the spectrum of  $H$  is *degenerate*. The number of the solutions corresponding to the eigenvalue  $E_0$  is called the *multiplicity of the degeneracy* of the level  $E_0$ .

If Equation 8.59 possesses a non-Abelian symmetry group, then the spectrum of the operator  $H$  is degenerate. The multiplicity of the degeneracy of the given level  $E_0$  is equal to the dimension of the unitary representation of the symmetry group (Demkov [1953]).

If the eigenvalues of Operator  $H$  are degenerate, then a non-Abelian symmetry group exists (Demkov [1953]).

### 8.5.1. THE HYDROGEN ATOM

(See Pauli [1926], Fock [1935], Bargmann [1936], Malkin and Manko [1966].) Setting  $n = 3$ ,  $A_i = 0$ ,  $V = -e/r$  in Equation 8.59, we can write the symmetry operators in the following form,

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}, \quad \mathbf{R} = \mathbf{p} \times \mathbf{L} - \mathbf{L} \times \mathbf{p} - 2e\mathbf{r}/r. \quad (8.63)$$

Vector  $\mathbf{R}$  is called the Runge-Lenz vector (Lenz [1924], see also the Laplace vector in [H1]). Operators  $L_i$  and  $R_j$  give rise to the non-Abelian algebra. That is why the spectrum of the hydrogen atom is degenerate.

### 8.5.2. ISOTROPIC HARMONIC OSCILLATOR

$$\mathbf{H} = \frac{1}{2} \sum_{k=1}^n [p_k^2 + \omega^2 x_k^2] = \omega \sum_{k=1}^n (c_k^+ c_k + 1/2). \quad (8.64)$$

Here  $c$  and  $c^+$  are the *creation and annihilation operators*

$$c_k = (2\omega)^{-1/2}(\omega x_k - ip_k), \quad c_k^+ = (2\omega)^{-1/2}(\omega x_k + ip_k),$$

satisfying the canonical commutation relations:

$$[c_i, c_k] = [c_i^+, c_k^+] = 0, \quad [c_i, c_k^+] = \delta_{ik}. \quad (8.65)$$

Symbol  $+$  means the Hermitian conjugation, therefore the unitary group of transformations  $\tilde{c}_k = U_k^s c_s$  is the symmetry group (Baker [1956]). Algebra of the symmetry operators is spanned by the following basis:

$$I_{jk} = c_j^+ c_k, \quad [I_{ij}, I_{kl}] = \delta_{jk} I_{il} - \delta_{il} I_{kj}, \quad [I_{ij}, \mathbf{H}] = 0. \quad (8.66)$$

This algebra was first found by Demkov [1953] in the case  $n = 3$ . He presented the following symmetry operators:  $L_{jk} = x_j p_k - x_k p_j$ ,  $H_{ij} = (1/\omega)(p_i p_j + \omega^2 x_i x_j)$ . It is obvious that

$$L_{jk} = i(c_j c_k - c_k^+ c_j), \quad H_{ij} = c_i^+ c_j + c_j^+ c_i + \delta_{ij}.$$

Existence of the algebra 8.66 implies the degeneracy in the spectrum.

### 8.5.3. TWO-DIMENSIONAL HARMONIC OSCILLATOR

$$\mathbf{H} = (1/2)(p_1^2 + p_2^2 + \omega_1^2 x_1^2 + \omega_2^2 x_2^2) \equiv (1/2) \sum_1^2 \omega_i (c_i c_i^+ + c_i^+ c_i). \quad (8.67)$$

Here the creation and annihilation operators

$$c_k = (2\omega_k)^{-1/2}(\omega_k x_k - ip_k), \quad c_k^+ = (2\omega_k)^{-1/2}(\omega_k x_k + ip_k)$$

satisfy the relations 8.65. Two operators  $N_i = c_i^+ c_i$  commute with  $\mathbf{H}$  for arbitrary  $\omega_i$ . Additional differential symmetry operators exist if and only if  $\omega_1 = l\omega$ ,  $\omega_2 = m\omega$ , where  $l$  and  $m$  are integers (Meshkov [1975]):

$$I_1 = (c_2^+)^l c_1^m + (c_1^+)^m c_2^l, \quad I_2 = i[(c_2^+)^l c_1^m - (c_1^+)^m c_2^l]. \quad (8.68)$$

The case  $l = 1$ ,  $m = 3$  was considered by Fokas and Lagerstrom [1980].

#### 8.5.4. TWO-DIMENSIONAL OSCILLATOR IN MAGNETIC FIELD

$$\mathbf{H} = (1/2)[(p_1 - bx_2)^2 + (p_2 + bx_1)^2 + \epsilon^2 r^2]. \quad (8.69)$$

Here  $b$  and  $\epsilon$  are any positive constants. In terms of the operators  $c_i, c_i^+$ :

$$\begin{aligned} 2ac_1^+ &= a^2 x_1 + p_2 + i(p_1 - a^2 x_2), \\ 2ac_1 &= a^2 x_1 + p_2 - i(p_1 - a^2 x_2), \\ 2ac_2^+ &= a^2 x_2 + p_1 + i(p_2 - a^2 x_1), \\ 2ac_2 &= a^2 x_2 + p_1 - i(p_2 - a^2 x_1), \end{aligned} \quad (8.70)$$

satisfying Equations 8.65, Hamiltonian 8.69 takes form 8.67, where  $\omega_1 = a^2 + b$ ,  $\omega_2 = a^2 - b$ ,  $a = (b^2 + \epsilon^2)^{1/4}$  (Dulock and McIntosh [1966]). Therefore there are only two differential symmetry operators  $N_i = c_i^+ c_i$  for any  $b$  and  $\epsilon$  ( $L_{12} = N_1 - N_2$ ). The additional symmetry operators exist and take form 8.68 if and only if  $\omega_1/\omega_2 = l/m$ , or  $\epsilon^2 = \omega^2 lm$ , and  $b = (\omega/2)(l - m)$ , where  $l$  and  $m$  are integers and  $\omega$  is an arbitrary number. Louck, Moshinsky, and Wolf [1973] constructed the pseudodifferential symmetry operators for Hamiltonian 8.69.

It is curious that two-dimensional Hamiltonian 8.69 is reduced to the one-dimensional  $\mathbf{H} = b(2c_1^+ c_1 + 1)$  when  $\epsilon = 0$ . In this case  $p_1 + bx_2$  and  $p_2 - bx_1$  are the symmetry operators (Landau [1930], Johnson and Lippmann [1949]). These operators are proportional to  $c_2^+ \pm c_2$ , i.e., they are the *hidden* variables in the terms  $c_i$ .

#### 8.5.5. THREE-DIMENSIONAL OSCILLATOR

$$\mathbf{H} = (1/2) \sum_{i=1}^3 \omega_i (c_i c_i^+ + c_i^+ c_i). \quad (8.71)$$

There exist three symmetry operators  $N_i = c_i^+ c_i$ , for any  $\omega_i$ . The extension of this algebra is possible in the following cases:

$$\begin{aligned}
 (a) \quad & \omega_1 = l\omega, \omega_2 = m\omega; I_1 = (c_1^+)^m c_2^l, I_2 = (c_2^+)^l c_1^m; \\
 (b) \quad & \omega_1 = l\omega, \omega_3 = k\omega; I_1 = (c_1^+)^k c_3^l, I_2 = (c_3^+)^l c_1^k; \\
 (c) \quad & \omega_2 = m\omega, \omega_3 = k\omega; I_1 = (c_3^+)^m c_2^k, I_2 = (c_2^+)^k c_3^m; \\
 (d) \quad & \omega_1 = l\omega, \omega_2 = m\omega, \omega_3 = k\omega; I_1 = (c_1^+)^m c_2^l, I_2 = (c_2^+)^l c_1^m, \\
 & I_3 = (c_3^+)^l c_1^k, I_4 = (c_1^+)^k c_3^l, I_5 = (c_3^+)^m c_2^k, I_6 = (c_2^+)^k c_3^m,
 \end{aligned} \tag{8.72}$$

where  $l, m, k$  are integers,  $\omega$  is an arbitrary number. Notice that in the case (d) dimension of the symmetry algebra is the same as for the isotropic oscillator. Quesne [1986] constructed the pseudodifferential symmetry operators for Hamiltonian 8.71.

### 8.5.6. THREE-DIMENSIONAL OSCILLATOR IN MAGNETIC FIELD

$$H = (1/2) \left[ (p_1 - bx_2)^2 + (p_2 + bx_1)^2 + p_3^2 + \epsilon^2 r^2 \right]. \tag{8.73}$$

Here  $b$  and  $\epsilon$  are any constants. In terms of the operators 8.70 and

$$c_3 = (2\epsilon)^{-1/2}(\epsilon x_3 - ip_3), \quad c_3^+ = (2\epsilon)^{-1/2}(\epsilon x_3 + ip_3).$$

Hamiltonian 8.73 takes the form 8.71, where  $\omega_1 = a^2 + b$ ,  $\omega_2 = a^2 - b$ ,  $\omega_3 = \epsilon$ ,  $a = (b^2 + \epsilon^2)^{1/4}$  (Quesne [1986]). There are three symmetry operators  $N_i = c_i^+ c_i$ , for any  $b$  and  $\epsilon$ . The extension of the symmetry algebra is possible in two cases (cf. Expressions 8.72):

$$\begin{aligned}
 (a) \quad & b = (\epsilon/2)(l - m)(lm)^{-1/2}, \omega_1 = l\omega, \omega_2 = m\omega, \\
 & \omega = \epsilon(lm)^{-1/2}; I_1 = (c_1^+)^m c_2^l, I_2 = (c_2^+)^l c_1^m;
 \end{aligned} \tag{8.74}$$

$$\begin{aligned}
 (b) \quad & b = (\epsilon/(2mn))(m^2 - n^2), \omega_1 = \epsilon m/n, \omega_2 = \epsilon n/m; \\
 & I_1 = (c_1^+)^{n^2} c_2^{m^2}, I_2 = (c_2^+)^{m^2} c_1^{n^2}, I_3 = (c_3^+)^m c_1^n, \\
 & I_4 = (c_1^+)^n c_3^m, I_5 = (c_3^+)^n c_2^m, I_6 = (c_2^+)^m c_3^n.
 \end{aligned} \tag{8.75}$$

Other examples of symmetries for quadratic Hamiltonians are given by Malkin and Manko [1979].

### 8.5.7. *N*-BODY PROBLEM

Let us mention in conclusion the one-dimensional systems of  $N$  pairwise interacting particles:

$$H = \left(\frac{1}{2}\right) \sum_{i=1}^N p_i^2 + \sum_{i<j}^N g_{ij} V(x_i - x_j).$$

Here  $g_{ij}$  are the constants and the potential takes one of the following forms:

$$V(x) = x^{-2}, \sinh^{-2} x, \sin^{-2} x, x^{-2} + \omega^2 x^2, P(x), \exp x,$$

$P$  is the Weierstrass function. There exist the complete sets of the pairwise commuting symmetry operators for some special  $g_{ij}$ . There are a lot of papers on this theme, therefore we refer only one review by Olshanetsky and Perelomov [1983].

## 8.6. MAXWELL EQUATIONS IN VACUUM

### 8.6.1. TENSOR FORMULATION

Let  $F_{ij} = -F_{ji}$  be the field tensor, then the source-free Maxwell equations are

$$\partial F^{ij} / \partial x^j = 0, \quad \partial \tilde{F}^{ij} / \partial x^j = 0, \quad (8.76)$$

where  $\tilde{F}_{ij} = (1/2)\epsilon_{ijkl}F^{kl}$  is the *dual tensor* and  $\epsilon_{ijkl}$  is the *Levi-Civita symbol*. Other notation are given in Section 8.1. The maximal Lie algebra of the symmetry vector fields is spanned by the 17 vector fields (Ibragimov [1968a]):

$$X_1 = \xi^i \frac{\partial}{\partial x^i} + \sum_{k<l} \left( \frac{\partial \xi^s}{\partial x^l} F_{sk} - \frac{\partial \xi^s}{\partial x^k} F_{sl} \right) \frac{\partial}{\partial F_{kl}}, \quad (8.77)$$

$$X_2 = \sum_{k<l} F_{kl} \frac{\partial}{\partial F_{kl}}, \quad X_3 = \sum_{k<l} \tilde{F}_{kl} \frac{\partial}{\partial F_{kl}}, \quad (8.78)$$

where  $\xi^i$  is the Killing vector of the conformal group:

$$\xi_i = 2x_i(cx) - (xx)c_i + bx_i + a_{ij}x^j + d_i. \quad (8.79)$$

Arbitrariness of the parameters  $a$  and  $b^i$  implies the following conserved currents:

$$I_i^k = \sigma_{ij} F^{jk} + F_{ij} \sigma^{jk} + \frac{1}{2} \delta_i^k F^{mn} \sigma_{mn}, \quad \Theta^k = x^i I_i^k. \quad (8.110)$$

If, for example,  $\sigma_{ij} = F_{ij}$ , then  $I_i^k = T_i^k$ ,  $\Theta^k = D^k$ , but if  $\sigma_{ij} = \tilde{F}_{ij}$ , then  $I_i^k = 0$ . If  $\sigma_{ij} = F_{ij,l}$ , then  $I_{il}^k = \partial T_i^k / \partial x^l$ , etc. Expressions 8.110 give all zilch-currents and some new currents. If we set  $\sigma = XF$ , where  $X$  is a polynomial of Operators 8.83, then Currents 8.110 are the higher conserved currents. Moreover replacing  $F_{ij}$  into any Lie-Bäcklund vector field  $\omega_{ij}$  in 8.110, we can obtain new conserved currents (Krivsky and Simulik [1989]).

## 8.7. DIRAC EQUATION

$$[\gamma^i P_i - m] \psi = 0. \quad (8.111)$$

Here  $m > 0$  is a parameter,  $\gamma^i$  are the *Dirac matrices*. Other notation are given in the beginning of Section 8.1. In standard representation  $\gamma^i$  take the following form

$$\gamma^0 = \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \quad i = 1, 2, 3, \quad (8.112)$$

where  $E$  is the unit  $2 \times 2$  matrix and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

are the *Pauli matrices*. The set of the following matrices

$$E, \quad \gamma^i, \quad \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{i}{4!} \epsilon_{jklm} \gamma^j \gamma^k \gamma^l \gamma^m,$$

$$\gamma_5 \gamma^i, \quad \sigma_{jk} = \frac{i}{2} [\gamma_j, \gamma_k],$$

where  $\gamma_i = \eta_{ij} \gamma^j$ , is the basis in the space of  $4 \times 4$  matrices. Let  $\epsilon_{jklm}$  be the Levi-Civita symbol,  $\epsilon_{0123} = 1$  and  $\epsilon_{jkl} = \epsilon_{0jkl}$ . Denoting  $\sigma_{k0} = i\alpha_k$ ,  $\sigma_{jk} = \epsilon_{jkl} \Sigma_l$ ,  $j, k, l > 0$ , one can write the basis matrices in the following explicit form:

$$\begin{aligned} \gamma_5 &= \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix}, \quad \gamma_5 \gamma^i = \begin{pmatrix} -\sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}, \quad i > 0, \\ \gamma_5 \gamma^0 &= \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}, \quad \alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}, \quad \Sigma_k = \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}. \end{aligned} \quad (8.113)$$

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## **Part I**

# **Apparatus of Group Analysis**

## ***Lie-Bäcklund Symmetries: Representation by Formal Power Series***

---

This chapter<sup>1</sup> describes the general structure of Lie-Bäcklund transformation groups. The emphasis in Sections 1.1 and 1.2 is on the role of the space  $\mathcal{A}$  of differential functions in the theory. Section 1.3 contains the main theorems and algorithms used for computation of Lie-Bäcklund symmetries of differential equations, while in Section 1.4 we present the recent application of formal power series to the construction of invariants.

We follow the internal logic of the subject. For a historical survey of the development of the theory of Lie-Bäcklund transformation groups starting with the works of Lie and Bäcklund and ending with the modern contributions, the reader is referred to [H1], Chapter 5 (see also Chapter 6, Section 6.2.1 in this volume, and Anderson and Ibragimov [1979]). A detailed discussion of the theory and many applications are to be found in Ibragimov [1983].

Lie-Bäcklund transformation groups and their invariants are represented by formal series. The question of the convergence of these series is not treated here. The reader who is interested in this aspect of the problem is referred to the classical book by Hardy [1949] and for a modern approach see Flato, Pinczon, and Simon [1977].

---

### **1.1. The universal space of modern group analysis**

The prolongation theory of Lie point and Lie contact transformation groups requires the introduction of functions depending not only on the independent variables  $x$  and dependent variables  $u$ , but also on derivatives of finite orders.

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This prolongation is sufficient in the context of classical Lie theory. However, it is insufficient for the natural generalization of the classical theory given by Lie-Bäcklund transformation groups. In this generalization, one deals with transformations acting on intrinsically infinite-dimensional spaces. This new approach mandates the space  $\mathcal{A}$  of differential functions as the universal space of modern group analysis.

### 1.1.1. The space $\mathcal{A}$

The space  $\mathcal{A}$  of differential functions was previously discussed in this Handbook (see [H1], Section 5.2, and [H2], Section 1.2.1). But, for the convenience of the reader, we reproduce here the necessary notation.

Let<sup>2</sup>

$$x = \{x^i\}, \quad u = \{u^\alpha\}, \quad u_{(1)} = \{u_i^\alpha\}, \quad u_{(2)} = \{u_{ij}^\alpha\}, \dots, \quad (1.1)$$

where  $\alpha = 1, \dots, m$ ;  $i, j = 1, \dots, n$ . These variables are connected by the total differentiations

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \dots, \quad i = 1, \dots, n, \quad (1.2)$$

as follows:

$$u_i^\alpha = D_i(u^\alpha), \quad u_{ij}^\alpha = D_j(u_i^\alpha) = D_j D_i(u^\alpha), \dots \quad (1.3)$$

The variables  $x$  are called *the independent variables*, and the variables  $u$  are known as *the differential variables* with the successive *derivatives*  $u_{(1)}, u_{(2)}, \dots$ . In the particular case, when  $n = m = 1$ , these derivatives will be denoted by  $u_1, u_2, \dots$ , so that  $u_1 = u_x, u_2 = u_{xx}$ , etc.

**DEFINITION 1.1.** A locally analytic function (i.e., locally expandable in a Taylor series with respect to all arguments) of a finite number of the variables 1.1 is called a differential function. The highest order of derivatives appearing in the differential function is called the order of this function. The vector space of all differential functions of finite order is denoted by  $\mathcal{A}$ .

The space  $\mathcal{A}$  has the intrinsic property of being closed under the derivation given by the total derivatives  $D_i$ .

<sup>2</sup>In [H1] and [H2], a slightly different notation for derivatives was used, namely,  $u_1, u_2$  instead of  $u_{(1)}, u_{(2)}$ , etc.

### 1.1.2. Lie-Bäcklund operators

**DEFINITION 1.2.** A differential operator of the form

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha} + \zeta_{i_1 i_2}^\alpha \frac{\partial}{\partial u_{i_1 i_2}^\alpha} + \dots, \quad (1.4)$$

where  $\xi^i, \eta^\alpha \in \mathcal{A}$ , and

$$\zeta_i^\alpha = D_i(\eta^\alpha - \xi^j u_j^\alpha) + \xi^j u_{ij}^\alpha, \quad \zeta_{i_1 i_2}^\alpha = D_{i_1} D_{i_2}(\eta^\alpha - \xi^j u_j^\alpha) + \xi^j u_{j i_1 i_2}^\alpha, \dots \quad (1.5)$$

is called a Lie-Bäcklund operator. The abbreviated operator

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} \quad (1.6)$$

is also referred to as a Lie-Bäcklund operator provided that its prolonged action given by Formulas 1.4 and 1.5 is implied.<sup>3</sup>

Operator 1.4 is a formal sum. However, it truncates when acting on any differential function. Hence, the action of Lie-Bäcklund operators is defined on the space  $\mathcal{A}$ .

### 1.1.3. Lie-Bäcklund algebra $L_B$

Let

$$X_v = \xi_v^i \frac{\partial}{\partial x^i} + \eta_v^\alpha \frac{\partial}{\partial u^\alpha}, \quad v = 1, 2,$$

be two Lie-Bäcklund operators 1.6. Their commutator (Lie bracket) is defined by the usual formula:

$$[X_1, X_2] = X_1 X_2 - X_2 X_1.$$

**THEOREM 1.1.** The commutator  $[X_1, X_2]$  is the Lie-Bäcklund operator given by

$$[X_1, X_2] = (X_1(\xi_2^i) - X_2(\xi_1^i)) \frac{\partial}{\partial x^i} + (X_1(\eta_2^\alpha) - X_2(\eta_1^\alpha)) \frac{\partial}{\partial u^\alpha} + \dots, \quad (1.7)$$

<sup>3</sup>Since the prolongations of Lie group generators and Lie-Bäcklund operators are uniquely determined by the prolongation formulas 1.5, there is no necessity to use a special notation for Operator 1.4 to indicate the prolongation.

where the terms denoted by dots are obtained by prolonging the coefficients of  $\partial/\partial x^i$  and  $\partial/\partial u^\alpha$  in accordance with Equations 1.4 and 1.5.

As a consequence, the following statement is valid.

**THEOREM 1.2.** *The set of all Lie-Bäcklund operators is an infinite dimensional Lie algebra with respect to the Lie bracket 1.7. We call it the Lie-Bäcklund algebra and denote it by  $L_B$ .*

#### 1.1.4. Properties of $L_B$ . Canonical operators

The Lie-Bäcklund algebra is endowed with the following properties (see, e.g., Ibragimov [1983]):

**I.** The total derivation 1.2 is a Lie-Bäcklund operator, i.e.,  $D_i \in L_B$ . Furthermore,

$$X_* = \xi_*^i D_i \in L_B \quad (1.8)$$

for any  $\xi_*^i \in \mathcal{A}$ .

**II.** Let  $L_*$  be the set of all Lie-Bäcklund operators of the form 1.8. Then  $L_*$  is an ideal of  $L_B$ , i.e.,  $[X, X_*] \in L_*$  for any  $X \in L_B$ . Indeed,

$$[X, X_*] = (X(\xi_*^i) - X_*(\xi^i)) D_i \in L_*.$$

**III.** In accordance with Property II, two operators  $X_1, X_2 \in L_B$  are said to be *equivalent* (i.e.,  $X_1 \sim X_2$ ) if  $X_1 - X_2 \in L_*$ . In particular, every operator  $X \in L_B$  is equivalent to an operator 1.4 with  $\xi^i = 0, i = 1, \dots, n$ . Namely,

$$X \sim X - \xi^i D_i \equiv (\eta^\alpha - \xi^i u_i^\alpha) \frac{\partial}{\partial u^\alpha} + \dots \quad (1.9)$$

**DEFINITION 1.3.** *Operators of the form*

$$X = \eta^\alpha \frac{\partial}{\partial u^\alpha} + \dots, \quad \eta^\alpha \in \mathcal{A}, \quad (1.10)$$

*are called canonical Lie-Bäcklund operators.*

Thus, Property III can be formulated as follows.

**THEOREM 1.3.** *Any operator  $X \in L_B$  is equivalent to a canonical Lie-Bäcklund operator.*

Canonical operators leave invariant the independent variables  $x^i$ . Therefore, the use of the canonical form is convenient, e.g., for investigating symmetries of integro-differential equations and for finding recursion operators (cf. Chapters 4 and 5; see also Section 1.2.5). It is important to note that the canonical form for the generator of a Lie point (or contact) transformation group is, in general, intrinsically a Lie-Bäcklund operator. Therefore, the canonical form was not used in classical Lie theory.

**IV.** The following statements describe all Lie-Bäcklund operators equivalent to generators of Lie point and Lie contact transformation groups.

**THEOREM 1.4.** *Operator 1.4 is equivalent to the infinitesimal operator of a one-parameter point transformation group if and only if its coordinates assume the form*

$$\xi^i = \xi_1^i(x, u) + \xi_*^i, \quad \eta^\alpha = \eta_1^\alpha(x, u) + (\xi_2^i(x, u) + \xi_*^i)u_i^\alpha,$$

where  $\xi_*^i \in \mathcal{A}$  is an arbitrary differential function and  $\xi_1^i, \xi_2^i, \eta_1^\alpha$  are arbitrary functions of  $x$  and  $u$ .

**THEOREM 1.5.** *Let  $m = 1$ . Then Operator 1.4 is equivalent to the infinitesimal operator of a one-parameter contact transformation group if and only if its coordinates assume the form*

$$\xi^i = \xi_1^i(x, u, u_{(1)}) + \xi_*^i, \quad \eta = \eta_1(x, u, u_{(1)}) + \xi_*^i u_i,$$

where  $\xi_*^i \in \mathcal{A}$  is an arbitrary differential function and  $\xi_1^i, \eta_1$  are arbitrary functions of  $x, u$  and  $u_{(1)}$ .

### EXAMPLES

To illustrate the notion of the canonical Lie-Bäcklund operators, we give here the generators of familiar point transformation groups written in both the classical and the canonical Lie-Bäcklund forms.

*Translations.* Let  $n = m = 1, u_1 = u_x$ . The generator of the group of translations along the  $x$ -axis,

$$X = \frac{\partial}{\partial x},$$

is shifted, by Equation 1.9, into the following canonical Lie-Bäcklund operator:

$$u_x \frac{\partial}{\partial u}.$$

*Dilations.* Let  $x, y$  be the independent variables, and  $k, c = \text{const.}$  The generator of non-homogeneous dilations and its canonical Lie-Bäcklund form are as follows:

$$X = x \frac{\partial}{\partial x} + ky \frac{\partial}{\partial y} + cu \frac{\partial}{\partial u} \sim (cu - xu_x - ky u_y) \frac{\partial}{\partial u} + \dots$$

*Galilean Boosts.* Here,

$$X = t \frac{\partial}{\partial x} + \frac{\partial}{\partial u} \sim (1 - tu_x) \frac{\partial}{\partial u} + \dots,$$

where  $t, x$  are the independent variables.

## 1.2. Formal transformation groups

For the sake of brevity, we introduce the sequence

$$z = (x, u, u_{(1)}, u_{(2)}, \dots) \quad (1.11)$$

with elements  $z^\nu, \nu \geq 1$ . Here, all that is essential is that some ordering be established within the sets  $x, u, u_{(1)}, \dots$ . For  $x$  and  $u$  we will use the natural ordering so that

$$z^i = x^i, \quad 1 \leq i \leq n, \quad z^{n+\alpha} = u^\alpha, \quad 1 \leq \alpha \leq m.$$

Denote by  $[z]$  any finite subsequence of  $z$ . Then elements of  $\mathcal{A}$  are written as  $f([z])$ .

### 1.2.1. The representation space $[[\mathcal{A}]]$

Consider formal power series in one symbol  $a$ :

$$f(z, a) = \sum_{k=0}^{\infty} f_k([z]) a^k, \quad f_k([z]) \in \mathcal{A}. \quad (1.12)$$

Let  $f$  and  $g$  be formal power series, where  $f$  is defined by Formula 1.12 and  $g$  is given by a similar formula, viz.

$$g(z, a) = \sum_{k=0}^{\infty} g_k([z])a^k, \quad g_k([z]) \in \mathcal{A}.$$

Their linear combination  $\lambda f([z]) + \mu g([z])$  with constant coefficients  $\lambda, \mu$  and product  $f([z]) \cdot g([z])$ , respectively, are defined as follows:

$$\lambda \sum_{k=0}^{\infty} f_k([z])a^k + \mu \sum_{k=0}^{\infty} g_k([z])a^k = \sum_{k=0}^{\infty} (\lambda f_k([z]) + \mu g_k([z]))a^k, \quad (1.13)$$

and

$$\left( \sum_{p=0}^{\infty} f_p([z])a^p \right) \cdot \left( \sum_{q=0}^{\infty} g_q([z])a^q \right) = \sum_{k=0}^{\infty} \left( \sum_{p+q=k} f_p([z])g_q([z]) \right) a^k. \quad (1.14)$$

The space of all formal power series 1.12 endowed with the addition and multiplication operations defined by Equations 1.13 and 1.14, respectively, is denoted by  $[[\mathcal{A}]]$ .

Lie point and Lie contact transformations, together with their prolongations of all orders, are represented by elements of the space  $[[\mathcal{A}]]$ . The utilization of this space is *a fortiori* necessary in the theory of Lie-Bäcklund transformation groups. Therefore, we call  $[[\mathcal{A}]]$  the representation space of modern group analysis.

### 1.2.2. Lie-Bäcklund equations

**DEFINITION 1.4.** Given an Operator 1.4, the Lie-Bäcklund equations are

$$\begin{aligned} \frac{d}{da} \bar{x}^i &= \xi^i([\bar{z}]), & \frac{d}{da} \bar{u}^\alpha &= \eta^\alpha([\bar{z}]), \\ \frac{d}{da} \bar{u}_i^\alpha &= \zeta_i^\alpha([\bar{z}]), & \frac{d}{da} \bar{u}_{ij}^\alpha &= \zeta_{ij}^\alpha([\bar{z}]), \dots, \end{aligned} \quad (1.15)$$

where  $\alpha = 1, \dots, m$  and  $i, j, \dots = 1, \dots, n$ .

In the case of canonical operators 1.10, the infinite system of equations 1.15 simplify to the finite system:

$$\frac{d}{da} \bar{u}^\alpha = \eta^\alpha([\bar{z}]), \quad \alpha = 1, \dots, m. \quad (1.16)$$

Then the transformations of the successive derivatives are obtained by the total differentiation:

$$\bar{u}_i^\alpha = D_i(\bar{u}^\alpha), \quad \bar{u}_{ij}^\alpha = D_i D_j(\bar{u}^\alpha), \dots, \quad (1.17)$$

### 1.2.3. Definition of a formal group

Consider sequences of elements

$$f^\nu(z, a) \in [[\mathcal{A}]], \quad \nu \geq 1,$$

such that

$$f^\nu(z, a) = \sum_{k=0}^{\infty} f_k^\nu([z])a^k, \quad f_0^\nu([z]) = z^\nu, \quad \nu = 1, 2, \dots, \quad (1.18)$$

where  $f_k^\nu([z])$ ,  $k = 1, 2, \dots$ , are arbitrary elements of  $\mathcal{A}$ .

**DEFINITION 1.5.** Define a transformation of Sequences 1.11 by

$$\bar{z}^\nu = f^\nu(z, a), \quad \nu \geq 1. \quad (1.19)$$

**DEFINITION 1.6.** Transformation 1.19 is called a formal one-parameter group if the coefficients in the formal series 1.18 obey the property:

$$f_k^\nu([\bar{z}]) = \sum_{l=0}^{\infty} \frac{(k+l)!}{k!l!} f_{k+l}^\nu([z])a^l, \quad \nu \geq 1, \quad k = 0, 1, 2, \dots. \quad (1.20)$$

**THEOREM 1.6.** Equations 1.20 are equivalent to the usual group property:

$$f^\nu(\bar{z}, b) = f^\nu(z, a + b), \quad \nu \geq 1, \quad (1.21)$$

written for formal power series.

For the proof, see Ibragimov [1983], Section 15.1.

This theorem requires more explanation. By definition, the left hand sides of Equation 1.21 are formal power series in one symbol  $b$ :

$$f^\nu(\bar{z}, b) = \sum_{l=0}^{\infty} f_l^\nu([\bar{z}])b^l.$$

In these series,  $\bar{z}$  is given by power series in one symbol  $a$ , hence after rearrangement we can write them as formal power series in two symbols  $a$  and  $b$ :

$$\sum_{l=0}^{\infty} f_l^v([\bar{z}])b^l = \sum_{k,l=0}^{\infty} g_{kl}^v([\bar{z}])a^k b^l. \quad (1.22)$$

Equation 1.21 asserts that the right hand side of Equation 1.22 can be rewritten as a formal power series in one symbol  $(a + b)$ , viz.

$$\sum_{k=0}^{\infty} f_k^v([\bar{z}])a^k = f^v(z, a + b), \quad v = 1, 2, \dots$$

**REMARK.** The group property, in its most general form

$$f^v(\bar{z}, b) = f^v(z, \phi(a, b)), \quad v \geq 1,$$

with the group composition law  $\phi(a, b)$ , can be transformed to Equation 1.21 by a suitable change of the group parameter  $a$ . The proof is similar to the classical case.

#### 1.2.4. Integration of Lie-Bäcklund equations

Consider Equations 1.15 with the initial conditions given in Section 1.2.3:

$$\begin{aligned} \frac{d}{da} \bar{x}^i &= \xi^i([\bar{z}]), \quad \bar{x}^i|_{a=0} = x^i, \\ \frac{d}{da} \bar{u}^\alpha &= \eta^\alpha([\bar{z}]), \quad \bar{u}^\alpha|_{a=0} = u^\alpha, \\ &\dots \end{aligned} \quad (1.23)$$

The formal integrability of the infinite system of Equations 1.23 is proved, e.g., in Ibragimov [1983], Section 15.1. It is also discussed in [H1], Section 5.2. However, for completeness we restate the relevant theorem here.

**THEOREM 1.7.** *Lie-Bäcklund equations 1.23 have a solution in the space  $[[\mathcal{A}]]$ . The solution is unique and is given by a sequence of formal power series of the form*

$$\bar{x}^i = x^i + \sum_{k=0}^{\infty} A_k^i([\bar{z}])a^k, \quad A_k^i([\bar{z}]) \in \mathcal{A},$$

$$\bar{u}^\alpha = u^\alpha + \sum_{k=0}^{\infty} B_k^\alpha([z])a^k, \quad B_k^\alpha([z]) \in \mathcal{A}, \quad (1.24)$$

. . . . .

The coefficients  $A_k^i([z])$ ,  $B_k^\alpha([z])$ , ... satisfy the group property 1.20.

### 1.2.5. Exponential map

The question immediately arises as to how one calculates the coefficients in Series 1.24. The fundamental idea is provided by the exponential map in Lie group theory. For the generator  $X$  of a Lie transformation group of points  $x = (x^1, \dots, x^n)$ , the group transformation is given by

$$\bar{x}^i = \exp(aX)(x^i), \quad i = 1, \dots, n, \quad (1.25)$$

where

$$\exp(aX) = 1 + aX + \frac{a^2}{2!}X^2 + \frac{a^3}{3!}X^3 + \dots \quad (1.26)$$

For example, let's take, in the case  $n = 1$ , the generator

$$X = x^2 \frac{\partial}{\partial x}.$$

Equation 1.25 yields:

$$\bar{x} = x + ax^2 + a^2x^3 + \dots \quad (1.27)$$

Similarly and more generally, as in the case of Lie group theory, one can easily prove that the solution 1.24 to the Lie-Bäcklund equations 1.23 is given by the exponential map

$$\bar{x}^i = \exp(aX)(x^i), \quad \bar{u}^\alpha = \exp(aX)(u^\alpha), \quad \bar{u}_i^\alpha = \exp(aX)(u_i^\alpha), \dots, \quad (1.28)$$

where  $\exp(aX)$  is given by Equation 1.26 written for the Lie-Bäcklund operator 1.4.

As we discussed before, we can restrict our consideration to canonical operators 1.10. Then Equations 1.23 reduce to the finite system of equations

$$\frac{d}{da}\bar{u}^\alpha = \eta^\alpha([\bar{z}]), \quad \bar{u}^\alpha|_{a=0} = u^\alpha \quad (1.29)$$

supplemented by the transformation formulas 1.17. It follows that formal groups of Lie-Bäcklund transformations can be constructed by virtue of the following theorem.

**THEOREM 1.8.** *Given a canonical Lie-Bäcklund operator,*

$$X = \eta^\alpha \frac{\partial}{\partial u^\alpha} + \dots,$$

*the corresponding formal one-parameter group is represented by the series*

$$\bar{u}^\alpha = u^\alpha + a\eta^\alpha + \frac{a^2}{2!}X(\eta^\alpha) + \dots + \frac{a^n}{n!}X^{n-1}(\eta^\alpha) + \dots \quad (1.30)$$

*together with its differential consequences*

$$\bar{u}_i^\alpha = u_i^\alpha + aD_i(\eta^\alpha) + \frac{a^2}{2!}X(D_i(\eta^\alpha)) + \dots + \frac{a^n}{n!}X^{n-1}(D_i(\eta^\alpha)) + \dots, \quad (1.31)$$

*and*

$$\bar{u}_{i_1 \dots i_s}^\alpha = u_{i_1 \dots i_s}^\alpha + aD_{i_1} \dots D_{i_s}(\eta^\alpha) + \dots + \frac{a^n}{n!}X^{n-1}(D_{i_1} \dots D_{i_s}(\eta^\alpha)) + \dots \quad (1.32)$$

*for any  $s > 1$ .*

**PROOF.** This result can be proved directly without using the general exponential map 1.28. Indeed, because of the uniqueness of the solution, one needs to only show that the formal power series 1.30 satisfy Equations 1.29. By definition,  $d\bar{u}^\alpha/da$  is obtained by differentiating the series 1.30 term by term:

$$\frac{d\bar{u}^\alpha}{da} = \eta^\alpha + aX(\eta^\alpha) + \frac{a^2}{2!}X^2(\eta^\alpha) + \dots + \frac{a^{n-1}}{(n-1)!}X^{n-1}(\eta^\alpha) + \dots.$$

The right hand side of this equation is identical with the formal power expansion (with respect to  $a$ ) of the function  $\eta^\alpha([\bar{z}])$ , where  $[\bar{z}]$  is given by the series 1.30 to 1.32. At last, we note that the series 1.31 for the transformation of the first-order derivatives and the similar series 1.32 for the higher-order derivatives are obtained from the series 1.30 via Formulas 1.17. Indeed, since the total derivatives  $D_i$  commute with any canonical Lie-Bäcklund operator  $X$ , one has

$$D_i \exp(aX) = \exp(aX)D_i.$$

Hence, Equation 1.30 and Formulas 1.17 yield the equations 1.31 and 1.32.

### EXAMPLES

We illustrate an application of Theorem 1.8 by the following examples with  $n = m = 1$ .

**Example 1.** Let

$$X = u_1 \frac{\partial}{\partial u} + u_2 \frac{\partial}{\partial u_1} + \cdots.$$

We have here  $\eta = u_1$ . Therefore,

$$X(\eta) = u_2, \quad X^2(\eta) = u_3, \dots, \quad X^{n-1}(\eta) = u_n.$$

Hence, Transformation 1.30 of the corresponding formal group has the form

$$\bar{u} = u + \sum_{n=1}^{\infty} \frac{a^n}{n!} u_n.$$

**Example 2.** Let

$$X = u_p \frac{\partial}{\partial u} + u_{p+1} \frac{\partial}{\partial u_1} + \cdots. \quad (1.33)$$

Here,  $\eta = u_p$  with an arbitrary positive integer  $p$ . We have:

$$X(\eta) = u_{2p}, \quad X^2(\eta) = u_{3p}, \dots, \quad X^{n-1}(\eta) = u_{np}.$$

Hence, Transformation 1.30 is given by the following formal power series:

$$\bar{u} = u + \sum_{n=1}^{\infty} \frac{a^n}{n!} u_{np}. \quad (1.34)$$

**Example 3.** Let

$$X = u_1^2 \frac{\partial}{\partial u} + 2u_1 u_2 \frac{\partial}{\partial u_1} + \cdots.$$

Here,  $\eta = u_1^2$ . Successively, we find:

$$X(\eta) = 4u_1^2 u_2, \quad X^2(\eta) = 8(u_1^3 u_3 + 3u_1^2 u_2^2),$$

$$X^3(\eta) = 16(u_1^4 u_4 + 12u_1^3 u_2 u_3 + 12u_1^2 u_2^3), \dots$$

Hence, Transformation 1.30 has the form:

$$\begin{aligned}\bar{u} = & u + au_1^2 + 2a^2u_1^2u_2 + \frac{4}{3}a^3(u_1^3u_3 + 3u_1^2u_2^2) \\ & + \frac{2}{3}a^4(u_1^4u_4 + 12u_1^3u_2u_3 + 12u_1^2u_2^3) + \cdots.\end{aligned}$$

### 1.2.6. Representations: Lie versus Lie-Bäcklund

Let us begin with an example, e.g., the series 1.27. It is convergent for  $|ax| < 1$ . In this case, the group transformation can be represented by the following analytic function:

$$\bar{x} = \frac{x}{1 - ax}. \quad (1.35)$$

In fact, Equation 1.35 is arrived at directly by integrating the Lie equation

$$\frac{d\bar{x}}{da} = \bar{x}^2, \quad \bar{x}|_{a=0} = x. \quad (1.36)$$

For any given  $x$ , the solution of the Cauchy problem 1.36 exists for sufficiently small  $a$ , namely for  $|a| < 1/|x|$ . This solution is unique and is given by Equation 1.35.

This approach, Lie's approach, of solving the Lie equations leads one to local Lie groups of transformations represented by functions similar to Equation 1.35.

However, in light of the problems posed by solving the Lie-Bäcklund equations 1.15, we emphasize an alternative direction for developing representations by power series similar to Equation 1.27, namely, representation given by the exponential map 1.28. In this latter approach, one immediately arrives at the theory of transformation groups represented by *formal* power series. For instance, in our example we can consider the transformation 1.27 without any restriction on the group parameter  $a$ . Then the representation is given by a divergent series for  $|ax| \geq 1$ . This is the direction taken in the book by Ibragimov [1983]. The essential guiding principle is that formal power series representations merit in their own right. In fact, they are necessary to develop any comprehensive, sensible theory of Lie-Bäcklund transformation groups. This situation pertains because the convergence problem for these formal power series cannot be universally solved and must be treated separately for each type of Lie-Bäcklund operators.

We note that series representations can, in principle, be used for constructing representations by functions. For instance, Equation 1.27 can be rewritten as

$$\bar{x} = x(1 + ax + a^2x^2 + \cdots).$$

One recognizes the terms in the parentheses as the Taylor expansion of the function  $1/(1 - ax)$ .

In practice, in the case of Lie groups, it is easier to solve the Lie equations than it is to sum the exponential map. While, at the present stage of development, for Lie-Bäcklund transformations there is no difference in difficulty between solving the Lie-Bäcklund equations and employing the exponential map.

Finally, we remark that the origin of the fundamental difference between Lie and Lie-Bäcklund transformation groups is that the former are determined by ordinary differential equations (Lie equations) while the latter are determined by evolutionary partial differential equations (Lie-Bäcklund equations) for  $\bar{u} = \bar{u}(x, a)$ :

$$\frac{\partial}{\partial a} \bar{u}^\alpha = \eta^\alpha(x, \bar{u}, \bar{u}_{(1)}, \dots, \bar{u}_{(k)}), \quad \alpha = 1, \dots, m.$$

### EXAMPLE

Consider the canonical Lie-Bäcklund operator 1.33 from Example 2 of Section 1.2.5. The corresponding group transformations are given by the formal power series 1.34. These series converge in the disk  $|a| < r$ , if the sequence  $(u, u_1, u_2, \dots)$  satisfies the inequalities

$$|u_{pk}| \leq Ck!r^{-k}, \quad C = \text{const.}, \quad k = 0, 1, 2, \dots \quad (1.37)$$

Then the transformed sequence  $(\bar{u}, \bar{u}_1, \bar{u}_2, \dots)$  satisfies the same inequalities 1.37. It follows that the representation 1.34 gives a local group of transformations in the class of *entire functions*  $u(x)$  of order  $p/(p - 1)$  determined by the conditions 1.37 (see Ibragimov [1983], Section 16.2).

Titov [1990] generalized this result to Operators 1.10 such that their coordinates  $\eta^\alpha$  are linear functions of  $u, u_{(1)}, \dots, u_{(k)}, k > 1$ , with coefficients depending on  $x$ .

Furthermore, it is proved in Ibragimov [1977] that the ideal  $L_* \subset L_B$  (see Section 1.1.4, Property II) generates an infinite local group of transformations acting in the *scale of Banach spaces of locally analytic functions*  $u(x)$ . Hence, the convergence problem concerns the quotient algebra  $L_B/L_*$ . More precisely, it concerns only the canonical Lie-Bäcklund operators 1.10 with coordinates  $\eta^\alpha$  depending (nonlinearly) on derivatives  $u_{(k)}$  with  $k > 1$ .

## 1.3. Invariant differential equations

Let  $F \in \mathcal{A}$  be an arbitrary differential function of order  $k \geq 1$ , i.e.,  $F = F(x, u, u_{(1)}, \dots, u_{(k)})$ . Consider the  $k$ th-order differential equation(s) ( $F$  may be

vector valued):

$$F(x, u, u_{(1)}, \dots, u_{(k)}) = 0, \quad (1.38)$$

Denote by  $[F]$  the set of Sequences 1.11 determined by the infinite system of equations

$$[F]: \quad F = 0, \quad D_i F = 0, \quad D_i D_j F = 0, \dots \quad (1.39)$$

The set  $[F]$  is called *the extended frame* of the differential equation 1.38 (cf. [H2], Chapter 1).

### 1.3.1. Determining equation for Lie-Bäcklund symmetries

**DEFINITION 1.7.** Equation 1.38 admits a Lie-Bäcklund transformation group  $G$  if the extended frame 1.39 is invariant under  $G$ . Then  $G$  is also called a symmetry group of Equation 1.38.

Here, we present the infinitesimal criteria proved in Ibragimov [1983].

**THEOREM 1.9.** Let  $G$  be a Lie-Bäcklund transformation group with the generator  $X$ . Equation 1.38 admits  $G$  if and only if

$$XF|_{[F]} = 0, \quad XD_i F|_{[F]} = 0, \dots$$

This criterion requires an infinite number of infinitesimal tests. However, it can be simplified and reduced to a finite number of tests.

**LEMMA.** The equation

$$XF|_{[F]} = 0$$

yields the infinite series of equations

$$XD_i F|_{[F]} = 0, \quad XD_i D_j F|_{[F]} = 0, \dots$$

Thus, we have the following finite criterium for calculating Lie-Bäcklund symmetries of differential equations:

**THEOREM 1.10.** Equation 1.38 admits the formal group  $G$  generated by the Lie-Bäcklund operator 1.4,

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha} + \zeta_{i_1 i_2}^\alpha \frac{\partial}{\partial u_{i_1 i_2}^\alpha} + \dots,$$

if and only if

$$XF|_{[F]} = 0, \quad (1.40)$$

where  $|_{[F]}$  means evaluated on the extended frame  $[F]$  defined by the infinite system of Equations 1.39.

**DEFINITION 1.8.** Equation 1.40 is called the determining equation for Lie-Bäcklund symmetries of the differential equation 1.38.

**REMARK.** The main difference between calculating Lie and Lie-Bäcklund symmetries lies in the fact that in the Lie case one deals with generators whose coefficients  $\xi^i$  and  $\eta^\alpha$  are differential functions of order zero (point symmetries) and *one* (contact symmetries), while in the Lie-Bäcklund case one has no *a priori* restriction on the order. As a result of this situation, additional machinery is necessary (see Ibragimov [1983] and Chapter 4 of this volume; cf. also [H1], Section 5.1.3).

### 1.3.2. Trivial symmetries

In investigating the Lie-Bäcklund symmetries, it is useful to take into account the following properties of the determining equation 1.40:

**I.** The ideal  $L_* \subset L_B$  (see Section 1.1.4, Property II) is admitted by any system of differential equations. Therefore, in applications to differential equations, the quotient algebra  $L_B/L_*$  is the object of study.

Indeed, let  $X_* \in L_*$ . It has the form 1.8,

$$X_* = \xi_*^i D_i.$$

Therefore,  $X_*F = \xi_*^i D_i F$ . Hence,  $X_*$  satisfies the determining equation 1.40,

$$X_*F|_{[F]} = 0,$$

by virtue of the definition of the extended frame  $[F]$  by Equations 1.39.

**II.** Equation 1.38 admits any canonical Lie-Bäcklund operator 1.10,

$$X = \eta^\alpha \frac{\partial}{\partial u^\alpha} + \dots,$$

with arbitrary coordinates  $\eta^\alpha$  satisfying the condition

$$\eta^\alpha|_{[F]} = 0, \quad \alpha = 1, \dots, m. \quad (1.41)$$

Indeed, for a locally analytic  $\eta^\alpha$ , Equation 1.41 yields

$$\eta^\alpha = \eta_0^\alpha F + \eta_1^{\alpha,i} D_i F + \eta_2^{\alpha,ij} D_i D_j F + \cdots + \eta_k^{\alpha,i_1 \dots i_k} D_{i_1} \cdots D_{i_k} F, \quad k < \infty.$$

Hence, the determining equation 1.40 is satisfied. Here, the coefficients  $\eta_0^\alpha, \eta_1^{\alpha,i}, \eta_2^{\alpha,ij}, \dots, \eta_k^{\alpha,i_1 \dots i_k} \in \mathcal{A}$  are arbitrary differential functions regular on the extended frame of the differential equation 1.38. Note that these coefficients may depend on  $F$  and its derivatives. However, the regularity requires that this dependence must be such that the functions  $\eta^\alpha$  are determined, finite and locally analytic on  $[F]$ .

The general case of Lie-Bäcklund operators 1.4 such that

$$\xi^i|_{[F]} = 0, \quad \eta^\alpha|_{[F]} = 0,$$

reduces, by virtue of the equivalence relation 1.9, to a canonical operator satisfying the condition 1.41.

Lie-Bäcklund symmetries given by the properties I and II are naturally considered as *trivial* symmetries of a given differential equation 1.38. In group analysis we are interested in nontrivial symmetries only.

#### EXAMPLE

Consider an arbitrary  $p$ th-order scalar evolution equation in one space variable  $x$ :

$$u_t = f(x, u, u_1, \dots, u_p),$$

where  $u_1 = u_x$ , etc. For this equation, nontrivial Lie-Bäcklund symmetries can be taken in the form

$$X = \eta(t, x, u, u_1, \dots, u_k) \frac{\partial}{\partial u} + \cdots.$$

For instance, consider the group of translations in  $t$ . Its generator  $\partial/\partial t$  is equivalent to the following nontrivial Lie-Bäcklund symmetry:

$$X = u_t \frac{\partial}{\partial u} + \cdots$$

and can be rewritten, by using the differential equation under consideration, in the form

$$X = f(x, u, u_1, \dots, u_p) \frac{\partial}{\partial u} + \cdots.$$

## 1.4. Invariants

In the group analysis, invariants of Lie point symmetry groups are used, e.g., for an explicit representation of invariant solutions (Ovsiannikov [1962]). The prolongation theory allows one to construct invariant differential equations via differential invariants of a given group prolonged to the space  $\mathcal{A}$ .

Lie-Bäcklund transformation groups do not have enough invariants in the space  $\mathcal{A}$ . However, they have as many differential invariants as any one-parameter Lie point transformation group, if one allows formal sums of elements from  $\mathcal{A}$ .

### 1.4.1. Differential invariants in $\mathcal{A}$

**DEFINITION 1.9.** Let  $G$  be a formal one-parameter group of transformations 1.24 acting in the space of Sequences 1.11. Let  $F([z]) \in \mathcal{A}$  be a differential function of order  $k$ . Then  $F$  is called a differential invariant of order  $k$  for the Lie-Bäcklund group  $G$  if

$$F([\bar{z}]) = F([z])$$

for any  $z$  given by Sequence 1.11 and for its image  $\bar{z}$  under Transformation 1.24.

This definition subsumes Lie's differential invariants of all orders and generalizes the notion to Lie-Bäcklund groups.

**THEOREM 1.11.** The function  $F([z]) \in \mathcal{A}$  is a differential invariant of the Lie-Bäcklund group with the generator  $X$  if and only if

$$XF = 0.$$

The proof can be carried out using the exponential map 1.28 and is similar to the Lie case.

### 1.4.2. Differential invariants of the Maxwell group

Consider the evolutionary part of the system of Maxwell's equations, viz.

$$\frac{\partial \mathbf{E}}{\partial t} = \text{curl} \mathbf{B}, \quad \frac{\partial \mathbf{B}}{\partial t} = -\text{curl} \mathbf{E}. \quad (1.42)$$

We treat this system as a Lie-Bäcklund equation 1.16. The corresponding Lie-Bäcklund transformation group with the group parameter  $a = t$  is called the

*Maxwell group.* Its Lie-Bäcklund operator is:

$$X = \sum_{i=1}^3 \left[ (\text{curl} \mathbf{B})^i \frac{\partial}{\partial E^i} - (\text{curl} \mathbf{E})^i \frac{\partial}{\partial B^i} \right].$$

**THEOREM 1.12.** *The Maxwell group has the following basis of differential invariants of order 0 and 1 (Ibragimov [1983], Section 17.2):*

$$\mathbf{x} = (x^1, x^2, x^3), \quad \text{div} \mathbf{E}, \quad \text{div} \mathbf{B},$$

where  $\mathbf{x}$  denotes the three spatial coordinates. Furthermore,<sup>4</sup> all other (higher-order) differential invariants are functions of these basic invariants and of the successive derivatives of  $\text{div} \mathbf{E}$  and  $\text{div} \mathbf{B}$  with respect to  $x$ .

It follows that, e.g., equations

$$\text{div} \mathbf{E} = f(\mathbf{x}), \quad \text{div} \mathbf{B} = g(\mathbf{x}), \quad (1.43)$$

with arbitrary functions  $f$  and  $g$ , are invariant under the action of the Maxwell group. Hence, if Equations 1.43 are satisfied at time  $t = 0$ , then they are satisfied at all subsequent times  $t$ . Here, the physics dictates the choice. Note that with the choice  $f = g = 0$ , we come to Maxwell's equations in a vacuum.

### 1.4.3. Lack of differential invariants in the one-dimensional case

Let us discuss the one-dimensional case. Namely, consider transformation groups involving one independent variable  $x$  and one dependent (i.e., differential) variable  $u$  together with successive derivatives  $u_1, u_2, \dots$ , such that  $u_{i+1} = D(u_i)$ ,  $u_0 = u$ , where

$$D \equiv D_x = \frac{\partial}{\partial x} + u_1 \frac{\partial}{\partial u} + u_2 \frac{\partial}{\partial u_1} + \dots$$

It is well known that local Lie groups of transformations have an infinite number of differential invariants. For Lie-Bäcklund transformation groups the corresponding problem is open, if we consider invariants to be from the space  $\mathcal{A}$ . More precisely, the situation in the one-dimensional case is described by the following statement.

<sup>4</sup>N.H. Ibragimov, unpublished.

**THEOREM 1.13.** *Let*

$$X = \eta(x, u, u_1, \dots, u_l) \frac{\partial}{\partial u} + \dots$$

*be a proper Lie-Bäcklund operator (not a generator of a Lie point or contact transformation group). Then it has no differential invariants in  $\mathcal{A}$  (in the sense of Definition 1.9) except arbitrary functions of  $x$ .*

**PROOF.** The invariance of a differential function  $F(x, u, \dots, u_k) \in \mathcal{A}$  of order  $k$ , under the action of the operator  $X$  is determined by the equation

$$XF \equiv \eta \frac{\partial F}{\partial u} + D(\eta) \frac{\partial F}{\partial u_1} + D^2(\eta) \frac{\partial F}{\partial u_2} + \dots + D^k(\eta) \frac{\partial F}{\partial u_k} = 0. \quad (1.44)$$

The highest-order derivative involved in this equation is  $u_{k+l}$ . It appears in the expression  $D^k(\eta)$ . Hence, Equation 1.44 yields

$$\frac{\partial F}{\partial u_k} = 0.$$

Consequently,

$$\frac{\partial F}{\partial u_{k-1}} = 0, \dots, \frac{\partial F}{\partial u} = 0.$$

This implies  $F = F(x)$ . Hence,  $X$  has only one functionally independent invariant in the space  $\mathcal{A}$ , namely  $x$ .

#### 1.4.4. Formal invariants in $[[\mathcal{A}]]$ : Outline of the method

In this section, we consider again the one-dimensional case of the preceding section. We show that one has for Lie-Bäcklund transformation groups the same situation as for Lie point transformation groups if one considers an approximation of invariants by divergent power series with coefficients from the space  $\mathcal{A}$ . For a detailed discussion of the method sketched here, see Anderson and Ibragimov [1994].

**DEFINITION 1.10.** *Two Lie-Bäcklund operators,*

$$X_1 = \xi_1(x, u, u_1, \dots, u_k) \frac{\partial}{\partial x} + \eta_1(x, u, u_1, \dots, u_l) \frac{\partial}{\partial u} + \dots,$$

and

$$X_2 = \xi_2(y, v, v_1, \dots, v_r) \frac{\partial}{\partial y} + \eta_2(y, v, v_1, \dots, v_s) \frac{\partial}{\partial v} + \dots,$$

are said to be similar if there exists an invertible formal transformation in  $[[A]]$ , namely given by the formal power series in one symbol  $\varepsilon$ :

$$y = x + \varepsilon Y_1(x, u, u_1, \dots, u_{m_1}) + \varepsilon^2 Y_2(x, u, u_1, \dots, u_{m_2}) + \dots, \quad Y_i \in \mathcal{A}, \quad (1.45)$$

$$v = u + \varepsilon V_1(x, u, u_1, \dots, u_{n_1}) + \varepsilon^2 V_2(x, u, u_1, \dots, u_{n_2}) + \dots, \quad V_i \in \mathcal{A}, \quad (1.46)$$

such that  $X_1$ , written in the new variables  $y, v$ , coincide with  $X_2$ , i.e.,

$$X_1 \equiv X_1(y) \frac{\partial}{\partial y} + X_1(v) \frac{\partial}{\partial v} = X_2. \quad (1.47)$$

This definition subsumes the usual definition of similarity in Lie theory.

We shall use the well-known fact that the generator  $X$  of any one-parameter Lie transformation group is similar to the generator of translations:

$$X_1 = \frac{\partial}{\partial x}. \quad (1.48)$$

Consider two Lie-Bäcklund operators: the operator  $X_1$  given in the form 1.48 and a canonical Lie-Bäcklund operator

$$X_2 = \eta(y, v, v_1, \dots, v_s) \frac{\partial}{\partial v} + \dots, \quad (1.49)$$

where  $v_{i+1} = D_y(v_i)$ . The method for approximating invariants of the Lie-Bäcklund operator  $X_2$  consists of three steps.

*FIRST*, we introduce the operator

$$X_\varepsilon = (1 - \varepsilon) \frac{\partial}{\partial y} + \varepsilon X_2, \quad (1.50)$$

where  $X_2$  is the operator 1.49. The operator  $X_\varepsilon$  continuously connects the operators 1.48 and 1.49:

$$X_{\varepsilon=0} = X_1, \quad X_{\varepsilon=1} = X_2.$$

*SECOND*, we establish the following result on similarity:

**THEOREM 1.14.** *Operator 1.50 is similar to the operator 1.48.*

The proof is direct. We find the similarity transformations 1.45 and 1.46, by solving Equation 1.47, in particular

$$X_1(y) = 1 - \varepsilon, \quad X_1(v) = \varepsilon \eta(y, v, v_1, \dots, v_s). \quad (1.51)$$

*THIRD*, take the known basis of all differential invariants of Operator 1.48, viz.

$$u, u_1, u_2, \dots, \quad (1.52)$$

and subject them to the similarity transformation obtained by solving Equations 1.51.

As a result, one obtains invariants represented by elements of the space  $[[\mathcal{A}]]$ , in accordance with the following definition.

**DEFINITION 1.11.** *An element of  $[[\mathcal{A}]]$ , namely a formal power series in one symbol  $\varepsilon$ ,*

$$F(y, v, v_1, \dots; \varepsilon) = F_0 + \varepsilon F_1 + \varepsilon^2 F_2 + \dots$$

*is called an invariant of the formal group with the Lie-Bäcklund operator 1.50 if the formal series  $X_\varepsilon F$  vanishes:*

$$X_\varepsilon F = 0.$$

*Here  $F_0, F_1, F_2, \dots$  depend on finite numbers of the variables  $y, v, v_1, \dots$  and are elements of  $\mathcal{A}$ .*

#### 1.4.5. Example to Section 1.4.4

Consider the Lie-Bäcklund operator

$$X_2 = v_1 \frac{\partial}{\partial v} + \dots$$

In this case, Operator 1.50 has the form

$$X_\varepsilon = (1 - \varepsilon) \frac{\partial}{\partial y} + \varepsilon v_1 \frac{\partial}{\partial v} + \varepsilon v_2 \frac{\partial}{\partial v_1} + \varepsilon v_3 \frac{\partial}{\partial v_2} + \dots \quad (1.53)$$

We select “the simplest nontrivial” solution of Equations 1.51 at each step and obtain the following transformation:

$$y = (1 - \varepsilon)x + \varepsilon u,$$

$$v = u + \varepsilon x u_1 + \varepsilon^2[(2 - u_1)x u_1 + \frac{1}{2}x^2 u_2] \\ + \varepsilon^3[4x u_1 + 2x^2 u_2 - \frac{3}{2}x^2 u_1 u_2 + x u_1^3 - 4x u_1^2 + \frac{1}{3!}x^3 u_3] + \dots.$$

One can easily find the transformations of derivatives, e.g.,

$$v_1 = u_1 + \varepsilon[2u_1 + x u_2 - u_1^2] + \varepsilon^2[4u_1 - 4u_1^2 + 4x u_1 u_2 + \frac{1}{2}x^2 u_3 + u_1^3] + \dots,$$

$$v_2 = u_2 + \varepsilon x u_1 + \varepsilon^2[4u_2 - 3u_1 u_2 + x u_3] + \dots, \quad v_3 = u_3 + \dots.$$

The reason for the diminishing orders of  $\varepsilon$  with the ascending orders of the derivatives is to illustrate which powers of  $\varepsilon$  are required to invert the above transformations up to order  $o(\varepsilon^3)$ .

In this process, one finds

$$u = v - \varepsilon x v_1 + \frac{1}{2}\varepsilon^2 x^2 v_2 + o(\varepsilon^2),$$

which, together with the expression for  $y$ , yields the following transcendental expression for  $x$ :

$$y - (1 - \varepsilon)x = \varepsilon v - \varepsilon^2 x v_1 + \frac{1}{2}\varepsilon^3 x^2 v_2 + o(\varepsilon^3).$$

These approximate equations can be calculated with any degree of precision. The general result is as follows (Anderson and Ibragimov [1994]):

The Lie-Bäcklund operator 1.53,

$$X_\varepsilon = (1 - \varepsilon)\frac{\partial}{\partial y} + \varepsilon v_1 \frac{\partial}{\partial v} + \varepsilon v_2 \frac{\partial}{\partial v_1} + \varepsilon v_3 \frac{\partial}{\partial v_2} + \dots,$$

is similar to the generator of the group of translations 1.48,

$$X_1 = \frac{\partial}{\partial x},$$

in accordance with Theorem 1.14. The similarity map 1.45 – 1.46 is given by

$$y = (1 - \varepsilon)x + \varepsilon u, \quad u = \sum_{i=0}^{\infty} \frac{(-\varepsilon x)^i}{i!} v_i, \quad (1.54)$$

where  $v_{i+1} = D_y(v_i)$ ,  $v_0 = v$ .

With this result in hand, we can map the infinite number of differential invariants 1.52 of Operator 1.48 into equivalent ones for the Lie-Bäcklund operator  $X_\varepsilon$ . Thus,

**THEOREM 1.15.** *The Lie-Bäcklund operator 1.53 has a countable basis of invariants determined by functions*

$$u(y, v, v_1, v_2, \dots), u_1(y, v, v_1, v_2, \dots), u_2(y, v, v_1, v_2, \dots), \dots \quad (1.55)$$

where  $y, v, v_1, v_2, \dots$  are defined recursively via Equations 1.54 using the differentiation formula

$$D_x \equiv D_x(y)D_y = (1 - \varepsilon(1 - u_1))D_y. \quad (1.56)$$

Using this theorem, one can easily find, e.g., the invariant  $u$  with the second order of precision:

$$u = v - \varepsilon y v_1 + \varepsilon^2 (v v_1 - y v_1 + \frac{1}{2} y^2 v_2) + o(\varepsilon^2).$$

## 1.5. Bäcklund transformations for evolution equations

### 1.5.1. Definition

The majority of the Bäcklund transformations that appear in applications satisfy the following definition (Anderson and Ibragimov [1978], Fokas and Anderson [1979]).

**DEFINITION 1.12.** *Consider the following two evolution equations:*

$$u_t = F(x, u, u_1, \dots, u_n) \quad (1.57)$$

and

$$v_t = H(x, v, v_1, \dots, v_n). \quad (1.58)$$

Here,  $u_1, \dots, u_n$  and  $v_1, \dots, v_n$  are successive derivatives of  $u$  and  $v$  with respect to the spatial variable  $x$ . Equations 1.57 and 1.58 are said to be related by a Bäcklund transformation if there exists an ordinary differential equation

$$\Phi(x, u, u_1, \dots, u_r, v, v_1, \dots, v_s) = 0 \quad (1.59)$$

which is invariant under the Lie-Bäcklund transformations group given by the coupled evolution equations 1.57 and 1.58 (cf. Section 1.2.6), i.e., generated by the canonical Lie-Bäcklund operator

$$X = F(x, u, \dots, u_n) \frac{\partial}{\partial u} + H(x, v, \dots, v_n) \frac{\partial}{\partial v} + \dots \quad (1.60)$$

If  $F \equiv H$ , one says that Equation 1.59 defines an auto-Bäcklund transformation of Equation 1.57.

This definition naturally generalizes to systems of evolution equation. Namely, consider Equations 1.57 and 1.58 with vector-valued functions

$$u = (u^1, \dots, u^m), \quad v = (v^1, \dots, v^m)$$

and

$$F = (F^1, \dots, F^m), \quad H = (H^1, \dots, H^m).$$

Then a Bäcklund transformation is given by Definition 1.12 where Equation 1.59 is a system of  $m$  equations with a vector-valued function

$$\Phi = (\Phi^1, \dots, \Phi^m)$$

and Operator 1.60 has the form:

$$X = \sum_{\alpha=1}^m \left( F^\alpha(x, u, \dots, u_n) \frac{\partial}{\partial u^\alpha} + H^\alpha(x, v, \dots, v_n) \frac{\partial}{\partial v^\alpha} \right) + \dots$$

### 1.5.2. Illustrative examples

**Example 1.** The Burgers equation

$$u_t = uu_x + u_{xx}$$

and the heat equation

$$v_t = v_{xx}$$

are related by the following Bäcklund transformation (known as the Hopf-Cole transformation; see, e.g., [H1], p. 182):

$$v_x - \frac{1}{2}uv = 0.$$

**Example 2.** The Korteweg-de Vries equation

$$u_t = uu_x + u_{xxx}$$

and the modified Korteweg-de Vries equation

$$v_t = \frac{1}{6}v^2v_x - v_{xxx}$$

are related by the Bäcklund transformation (known as the Miura transformation; see, e.g., [H1], p. 189):

$$v_x - \gamma(u + v^2) = 0, \quad \gamma = \pm 1.$$

**Example 3.** The Korteweg-de Vries equation

$$u_t = u_{xxx} + 6uu_x$$

has the auto-Bäcklund transformation (see, e.g., [H1], p.189):

$$u_x + v_x + (u - v)(\lambda - 2(u + v))^{1/2} = 0, \quad \lambda = \text{const.}$$

**Example 4.** The classical example is the Bonnet equation

$$2u_{\xi\eta} = \sin(2u).$$

It is invariant under the Bianchi-Lie transformation<sup>5</sup>

$$u(\xi, \eta) \mapsto v(\xi, \eta)$$

determined by the system of first-order equations:

$$u_\xi + v_\xi = \sin(u - v), \quad u_\eta - v_\eta = \sin(u + v).$$

Bianchi-Lie transformation satisfies Definition 1.12 if we rewrite the Bonnet equation, in new independent variables  $t = \xi + \eta$ ,  $x = \xi - \eta$  and dependent variables  $u^1, u^2$ , as an evolutionary system:

$$u_t^1 = u^2, \quad u_t^2 = u_{xx}^1 + \sin u^1 \cos u^1.$$

<sup>5</sup>Historical remarks are to be found in Anderson and Ibragimov [1978].

Then the Bianchi-Lie transformation is an auto-Bäcklund transformation of this system and is given by the following two equations 1.59:

$$v_x^1 + u^2 - \sin u^1 \cos v^1 = 0, \quad u_x^1 + v^2 + \sin v^1 \cos u^1 = 0.$$

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## ***Approximate Transformation Groups and Deformations of Symmetry Lie Algebras***

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In applied sciences, differential equations depending on a small parameter are of frequent occurrence. Therefore, a theory based on the new concept of an approximate group was developed for tackling differential equations with a small parameter — particularly those relating to applications.

One-parameter approximate transformation groups were introduced in Baikov, Gazizov, and Ibragimov [1987a], [1988a], [1989a]. Recently, the theory was evolved in Baikov, Gazizov, and Ibragimov [1993], and Lie theorems were extended to multi-parameter approximate transformation groups.

Though the new concept maintains the essential features of the Lie group theory, it has certain peculiarities both in theory and applications.

This chapter provides a concise introduction to the theory of approximate transformation groups and *regular* approximate symmetries of differential equations with a small parameter.

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### **2.1. Preliminaries**

In what follows, all the functions under consideration are assumed to be locally analytic in their arguments.

#### **2.1.1. Notation**

Let  $z = (z^1, \dots, z^N)$  be an independent variable and  $\varepsilon$  a small parameter. We write

$$F(z, \varepsilon) = o(\varepsilon^p),$$

if

$$\lim_{\varepsilon \rightarrow 0} \frac{F(z, \varepsilon)}{\varepsilon^p} = 0,$$

or equivalently, if

$$F(z, \varepsilon) = \varepsilon^{p+1} \varphi(z, \varepsilon)$$

where  $\varphi(z, \varepsilon)$  is an analytic function defined in a neighborhood of  $\varepsilon = 0$  and  $p$  is a positive integer.

If

$$f(z, \varepsilon) - g(z, \varepsilon) = o(\varepsilon^p),$$

we write

$$f(z, \varepsilon) = g(z, \varepsilon) + o(\varepsilon^p),$$

or, briefly

$$f \approx g$$

when there is no ambiguity. In words,  $f$  is approximately equal to  $g$ .

In theoretical discussions, approximate equalities are considered with an error  $o(\varepsilon^p)$  of an arbitrary order  $p \geq 1$ . However, in most of the applications and examples (Sections 2.2.4, 2.2.5, 2.4.8, 2.5.3 to 2.5.6, 2.6.2, 2.7), the theory is simplified to the case  $p = 1$ .

### 2.1.2. Approximate Cauchy problem and its solution

An approximate Cauchy problem,

$$\frac{dz}{dt} \approx f(z, \varepsilon), \quad (2.1)$$

$$z|_{t=0} \approx \alpha(\varepsilon), \quad (2.2)$$

is defined as follows.

The approximate differential equation 2.1 denotes the class of differential equations

$$\frac{dz}{dt} = g(z, \varepsilon) \quad (2.3)$$

with functions  $g(z, \varepsilon) \approx f(z, \varepsilon)$ .

Similarly, the approximate initial condition 2.2 is the class of conditions

$$z|_{t=0} = \beta(\varepsilon) \quad (2.4)$$

with  $\beta(\varepsilon) \approx \alpha(\varepsilon)$ .

The following known statement on a continuous dependence of a parameter hints the natural definition of the solution to Problem 2.1 – 2.2 and furnishes with the theorem on the existence and uniqueness of the solution of an arbitrary approximate Cauchy problem.

**THEOREM 2.1.** *Let the functions  $f(z, \varepsilon)$ ,  $g(z, \varepsilon)$  be analytic in a neighborhood of a given point  $(z_0, 0)$ . Let*

$$f(z, \varepsilon) = g(z, \varepsilon) + o(\varepsilon^p),$$

and

$$\alpha(\varepsilon) = \beta(\varepsilon) + o(\varepsilon^p), \quad \alpha(0) = \beta(0) = z_0.$$

*Then there exist the solutions  $z(t, \varepsilon)$  and  $w(t, \varepsilon)$  of the problems*

$$\frac{dz}{dt} = f(z, \varepsilon), \quad z|_{t=0} = \alpha(\varepsilon),$$

and

$$\frac{dw}{dt} = g(w, \varepsilon), \quad w|_{t=0} = \beta(\varepsilon),$$

*respectively. These solutions are locally analytic, unique and approximately equal, viz.*

$$z(t, \varepsilon) = w(t, \varepsilon) + o(\varepsilon^p).$$

Thus, the solutions to all the problems of the form 2.3 – 2.4 coincide in a given precision. Therefore, *the solution of the approximate Cauchy problem 2.1 – 2.2 is defined as the class of functions  $z(t, \varepsilon)$  approximately equal to the solution of any specified problem 2.3 – 2.4.* According to the above theorem, this definition does not depend on the choice of Problem 2.3 – 2.4, and the solution of the approximate Cauchy problem 2.1 – 2.2 is determined uniquely. In applications, it is convenient to identify the solution of the approximate Cauchy problem with the solution of a specified problem 2.3 – 2.4.

### 2.1.3. Completely integrable systems

A system of approximate equations

$$\frac{\partial z^i}{\partial a^\alpha} \approx \psi_\alpha^i(z, a, \varepsilon), \quad i = 1, \dots, N, \quad \alpha = 1, \dots, r, \quad (2.5)$$

are said to be completely integrable, if

$$\frac{\partial}{\partial a^\beta} \left( \frac{\partial z^i}{\partial a^\alpha} \right) \approx \frac{\partial}{\partial a^\alpha} \left( \frac{\partial z^i}{\partial a^\beta} \right)$$

whenever  $z$  satisfies Equation 2.5.

The completely integrable system 2.5 with arbitrary initial conditions

$$z^i|_{a=0} \approx z_0^i$$

has the unique approximate solution of the form

$$z^i \approx f_0^i(a) + \varepsilon f_1^i(a) + \cdots + \varepsilon^p f_p^i(a), \quad i = 1, \dots, N.$$

---

## 2.2. One-parameter approximate groups

A detailed discussion of the material presented here is to be found in Baikov, Gazizov, and Ibragimov [1989a].

### 2.2.1. Definition

Consider a set of smooth vector-functions  $f_0(z, a), f_1(z, a), \dots, f_p(z, a)$  with coordinates

$$f_0^i(z, a), f_1^i(z, a), \dots, f_p^i(z, a), \quad i = 1, \dots, N.$$

Let us define the one-parameter family  $G$  of *approximate transformations*

$$\bar{z}^i \approx f_0^i(z, a) + \varepsilon f_1^i(z, a) + \cdots + \varepsilon^p f_p^i(z, a), \quad i = 1, \dots, N, \quad (2.6)$$

of points  $z = (z^1, \dots, z^N) \in R^N$  into points  $\bar{z} = (\bar{z}^1, \dots, \bar{z}^N) \in R^N$  as the class of invertible transformations

$$\bar{z} = f(z, a, \varepsilon) \quad (2.7)$$

with vector-functions  $f = (f^1, \dots, f^N)$  such that

$$f^i(z, a, \varepsilon) \approx f_0^i(z, a) + \varepsilon f_1^i(z, a) + \cdots + \varepsilon^p f_p^i(z, a).$$

Here,  $a$  is a real parameter, and the following condition is imposed:

$$f(z, 0, \varepsilon) \approx z.$$

Furthermore, it is assumed that Transformation 2.7 is defined for any value of  $a$  from a small neighborhood of  $a = 0$ , and that, in this neighborhood, the equation  $f(z, a, \varepsilon) \approx z$  yields  $a = 0$ .

The set  $G$  of Transformations 2.6 is called a (local) one-parameter approximate transformation group if

$$f(f(z, a, \varepsilon), b, \varepsilon) \approx f(z, a + b, \varepsilon)$$

for all Transformations 2.7.

Here,  $f$  does not necessarily denote the same function at each occurrence.

**Example.** Let us take  $N = 1$  and set  $z = x$ . Consider the following two functions:

$$f(x, a, \varepsilon) = x + a(1 + \varepsilon x + \frac{1}{2}\varepsilon a)$$

and

$$g(x, a, \varepsilon) = x + a(1 + \varepsilon x)(1 + \frac{1}{2}\varepsilon a).$$

They are equal in the first order of precision, viz.

$$g(x, a, \varepsilon) = f(x, a, \varepsilon) + \varepsilon^2 \varphi(x, a), \quad \varphi(x, a) = \frac{1}{2}a^2 x,$$

and satisfy the approximate group property. Indeed,

$$f(g(x, a, \varepsilon), b, \varepsilon) = f(x, a + b, \varepsilon) + \varepsilon^2 \phi(x, a, b, \varepsilon),$$

where

$$\phi(x, a, b, \varepsilon) = \frac{1}{2}a(ax + ab + 2bx + \varepsilon abx).$$

### 2.2.2. Approximate group generator

The generator of the approximate group  $G$  of Transformations 2.6 is the class of first-order linear differential operators

$$X = \xi^i(z, \varepsilon) \frac{\partial}{\partial z^i} \tag{2.8}$$

such that

$$\xi^i(z, \varepsilon) \approx \xi_0^i(z) + \varepsilon \xi_1^i(z) + \cdots + \varepsilon^p \xi_p^i(z),$$

where the vector fields  $\xi_0, \xi_1, \dots, \xi_p$  are given by

$$\xi_v^i(z) = \frac{\partial f_v^i(z, a)}{\partial a} \Big|_{a=0}, \quad v = 0, \dots, p; \quad i = 1, \dots, N.$$

In what follows, an approximate group generator is written as

$$X \approx \left( \xi_0^i(z) + \varepsilon \xi_1^i(z) + \dots + \varepsilon^p \xi_p^i(z) \right) \frac{\partial}{\partial z^i}.$$

It is written also in a specified form, viz.

$$X = \xi^i(z, \varepsilon) \frac{\partial}{\partial z^i} \equiv \left( \xi_0^i(z) + \varepsilon \xi_1^i(z) + \dots + \varepsilon^p \xi_p^i(z) \right) \frac{\partial}{\partial z^i}. \quad (2.9)$$

### 2.2.3. The approximate Lie equation

For approximate groups, the Lie theorem can be proved in the following modification.

**THEOREM 2.2.** *Let  $G$  be an approximate group given by Transformations 2.6, and let  $X$  be its generator 2.9. Then the function  $f(z, a, \varepsilon)$  satisfies the equation*

$$\frac{\partial f(z, a, \varepsilon)}{\partial a} \approx \xi(f(z, a, \varepsilon), \varepsilon).$$

*Conversely, for any Operator 2.9, the solution of the approximate Cauchy problem*

$$\frac{d\bar{z}}{da} \approx \xi(\bar{z}, \varepsilon), \quad (2.10)$$

$$\bar{z}|_{a=0} \approx z, \quad (2.11)$$

*determines an approximate one-parameter group of Transformations 2.6 with the group parameter  $a$ .*

Equation 2.10 is called *the approximate Lie equation*.

### 2.2.4. The first-order approximation to the Lie equation

Consider the approximate generator 2.9 with  $p = 1$ :

$$X = (\xi_0(z) + \varepsilon \xi_1(z)) \frac{\partial}{\partial z}.$$

The corresponding approximate group transformations have the form:

$$\bar{z} \approx f_0(z, a) + \varepsilon f_1(z, a).$$

In this case, the approximate Lie equation 2.10 is written:

$$\frac{d(f_0 + \varepsilon f_1)}{da} \approx \xi_0(f_0 + \varepsilon f_1) + \varepsilon \xi_1(f_0 + \varepsilon f_1).$$

It follows that the approximate Cauchy problem 2.10 – 2.11 reduces to the usual Cauchy problem of the form:

$$\frac{df_0}{da} = \xi_0(f_0), \quad \frac{df_1}{da} = \xi'_0(f_0)f_1 + \xi_1(f_0), \quad (2.12)$$

$$f_0|_{a=0} = z, \quad f_1|_{a=0} = 0. \quad (2.13)$$

Here,

$$\xi'_0(z) = \left\| \frac{\partial \xi_0^i(z)}{\partial z^j} \right\|.$$

In coordinates, the differential equations 2.12 are written:

$$\frac{df_0^i}{da} = \xi_0^i(f_0), \quad \frac{df_1^i}{da} = \sum_{k=1}^N \frac{\partial \xi_0^i(z)}{\partial z^k} \bigg|_{z=f_0} f_1^k + \xi_1^i(f_0).$$

**REMARK.** The case of an arbitrary  $p$  is treated in Baikov, Gazizov, and Ibragimov [1987a], [1988a], [1989a].

### 2.2.5. Solution of the approximate Lie equation in the first-order precision (examples)

**Example 1.** Let  $N = 1$ ,  $z = x$ , and let

$$X = (1 + \varepsilon x) \frac{\partial}{\partial x}.$$

Here,  $\xi_0(x) = 1$ ,  $\xi_1(x) = x$ , and the corresponding Cauchy problem 2.12 – 2.13 is written:

$$\frac{df_0}{da} = 1, \quad \frac{df_1}{da} = f_0,$$

$$f_0|_{a=0} = x, \quad f_1|_{a=0} = 0.$$

Its solution has the form

$$f_0 = x + a, \quad f_1 = xa + \frac{a^2}{2}.$$

Hence, the approximate group is given by (cf. the example in Section 2.2.1)

$$\bar{x} \approx x + a + \varepsilon \left( xa + \frac{a^2}{2} \right).$$

This approximate transformation is certainly contained in the Taylor expansion (in  $\varepsilon$ ) of the exact group transformations generated by  $X$ :

$$\bar{x} = x \exp(a\varepsilon) + \frac{\exp(a\varepsilon) - 1}{\varepsilon} = x + a + \varepsilon \left( xa + \frac{a^2}{2} \right) + \varepsilon^2 \left( \frac{a^2}{2}x + \frac{a^3}{6} \right) + \dots.$$

**Example 2.** Let  $N = 2$ ,  $z = (x, y)$ , and let

$$X = (1 + \varepsilon x^2) \frac{\partial}{\partial x} + \varepsilon xy \frac{\partial}{\partial y}.$$

Here,  $\xi_0(x, y) = (1, 0)$ ,  $\xi_1(x, y) = (x^2, xy)$ , and the Cauchy problem 2.12 – 2.13 is written:

$$\frac{df_0^1}{da} = 1, \quad \frac{df_0^2}{da} = 0, \quad \frac{df_1^1}{da} = (f_0^1)^2, \quad \frac{df_1^2}{da} = f_0^1 f_0^2,$$

$$f_0^1|_{a=0} = x, \quad f_0^2|_{a=0} = y, \quad f_1^1|_{a=0} = 0, \quad f_1^2|_{a=0} = 0.$$

The solution of this problem yields the following approximate group transformations:

$$\bar{x} \approx x + a + \varepsilon \left( x^2 a + xa^2 + \frac{a^3}{3} \right), \quad \bar{y} \approx y + \varepsilon \left( xya + \frac{ya^2}{2} \right).$$

The exact group transformations are

$$\bar{x} = \frac{\delta x \cos \delta a + \sin \delta a}{\delta (\cos \delta a - \delta x \sin \delta a)}, \quad \bar{y} = \frac{y}{\cos \delta a - \delta x \sin \delta a},$$

where  $\delta = \sqrt{\varepsilon}$ .

## 2.3. Approximate Lie algebras

Current developments in the theory of multi-parameter approximate groups require a modification of basic notions of Lie algebras. This section, based on Baikov, Gazizov, and Ibragimov [1993], is aimed to introduce the minimum of the necessary modifications.

### 2.3.1. Approximate operators

Let

$$\xi_0^i(z), \xi_1^i(z), \dots, \xi_p^i(z), \quad i = 1, \dots, N,$$

be a set of smooth vector fields. An *approximate operator* is the class of first-order differential operators (cf. Section 2.2.2),

$$X = \xi^i(z, \varepsilon) \frac{\partial}{\partial z^i}$$

such that

$$\xi^i(z, \varepsilon) \approx \xi_0^i(z) + \varepsilon \xi_1^i(z) + \dots + \varepsilon^p \xi_p^i(z), \quad i = 1, \dots, N.$$

According to this definition,  $X$  does not necessarily denote here the same operator at each occurrence. Thus, in approximate group theory, we deal with “unspecified” operators.

### 2.3.2. Approximate commutator

An approximate commutator of the operators  $X_1$  and  $X_2$  is an approximate operator denoted by  $[X_1, X_2]$  and given by

$$[X_1, X_2] \approx X_1 X_2 - X_2 X_1.$$

**Example.** For the operators

$$X_1 = \frac{\partial}{\partial x} + \varepsilon x \frac{\partial}{\partial y}, \quad X_2 = \frac{\partial}{\partial y} + \varepsilon y \frac{\partial}{\partial x},$$

the approximate commutator is equal to zero up to the error  $\varepsilon^2$ , while, in the second

order of precision, it is equal to

$$X_3 = \varepsilon^2 \left( x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} \right).$$

### 2.3.3. Properties of the commutator

The approximate commutator adopts the usual properties, viz.

a) the linearity:

$$[aX_1 + bX_2, X_3] \approx a[X_1, X_3] + b[X_2, X_3], \quad a, b = \text{const.},$$

b) the skew-symmetry:

$$[X_1, X_2] \approx -[X_2, X_1],$$

c) the Jacobi identity:

$$[[X_1, X_2], X_3] + [[X_2, X_3], X_1] + [[X_3, X_1], X_2] \approx 0.$$

### 2.3.4. Approximate Lie algebra of approximate operators

A vector space  $L$  of approximate operators is called *an approximate Lie algebra of operators*, if it is closed (in the approximation of a given order  $p$ ) under the approximate commutator, i.e., if

$$[X_1, X_2] \in L$$

for any  $X_1, X_2 \in L$ . Here the approximate commutator  $[X_1, X_2]$  is calculated to the precision indicated.

### 2.3.5. Linear independence

Approximate operators  $X_\alpha$ ,  $\alpha = 1, \dots, r$ , are said to be linearly independent if the approximate equation

$$C^\alpha X_\alpha \approx 0$$

with constants coefficients  $C^\alpha$  (they are assumed to be independent of  $\varepsilon$ ) yields

$$C^\alpha = 0, \quad \alpha = 1, \dots, r.$$

**Example.** The approximate operators

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \varepsilon \frac{\partial}{\partial t}$$

are linearly dependent in the zero order of precision, while they are linearly independent in the first order of precision.

### 2.3.6. Basis of an approximate Lie algebra

A basis of an approximate Lie algebra  $L$  is a set of linearly independent approximate operators that span the vector space  $L$ . If  $L$  has a finite basis consisting of  $r < \infty$  operators, then it is said to be finite-dimensional ( $r$ -dimensional) and is often denoted by  $L_r$ .

### 2.3.7. Essential operators

Let  $L$  be an approximate Lie algebra, and let  $\{X_\alpha\}$  be a set of linearly independent approximate operators from  $L$ , considered in a given approximation of order  $p$ . Denote by  $\{X'_\beta\}$  the set of approximate operators obtained by multiplying elements of  $\{X_\alpha\}$  by  $\varepsilon, \varepsilon^2, \dots, \varepsilon^p$  and neglecting the terms of order  $o(\varepsilon^p)$ . If the set of approximate operators

$$\{X_\alpha, X'_\beta\}$$

provides a basis of  $L$ , then the operators  $X_\alpha$  are said to be *essential operators* of the approximate Lie algebra  $L$ .

*An approximate Lie algebra is completely determined, in a given precision, by its essential operators.*

### 2.3.8. Structure constants of an approximate Lie algebra

Let  $L_r$  be an  $r$ -dimensional approximate Lie algebra, and let  $\{X_\alpha\}, \alpha = 1, \dots, r$ , be its basis. Then

$$[X_\alpha, X_\beta] \approx c_{\alpha\beta}^\gamma X_\gamma, \quad \alpha, \beta = 1, \dots, r,$$

with constant (and independent of  $\varepsilon$ ) coefficients  $c_{\alpha\beta}^\gamma, \alpha, \beta, \gamma = 1, \dots, r$ .

The coefficients  $c_{\alpha\beta}^\gamma$  are termed *the structure constants* of  $L_r$ .

### 2.3.9. Example of an approximate Lie algebra and its essential operators

Consider the following operators:

$$X_1 = \frac{\partial}{\partial x} + \varepsilon x \frac{\partial}{\partial y}, \quad X_2 = \frac{\partial}{\partial y} + \varepsilon y \frac{\partial}{\partial x},$$

$$X_3 = \varepsilon \left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right), \quad X_4 = \varepsilon \frac{\partial}{\partial x}, \quad X_5 = \varepsilon \frac{\partial}{\partial y}.$$

Their linear span is not a Lie algebra in the usual (exact) sense. For instance, the (exact) commutator

$$[X_1, X_2] = \varepsilon^2 \left( x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} \right)$$

is not a linear combination of the above operators.

However, these operators span an approximate Lie algebra in the first order of precision. Indeed, in this approximation, we have the following non-vanishing commutators:

$$[X_1, X_3] \approx -[X_3, X_1] \approx X_4, \quad [X_2, X_3] \approx -[X_3, X_2] \approx X_5.$$

Thus our operators span the 5-dimensional approximate Lie algebra  $L_5$  with the following non-zero structure constants:

$$c_{13}^4 = -c_{31}^4 = 1, \quad c_{23}^5 = -c_{32}^5 = 1.$$

The operators  $X_1$ ,  $X_2$ , and  $X_3$  are the essential operators of  $L_5$ .

## 2.4. Multi-parameter approximate groups

In this section we recount the results of current developments. For the proofs of the main theorems, the reader is referred to Baikov, Gazizov, and Ibragimov [1993].

### 2.4.1. Definition

Consider approximate transformations (cf. Section 2.2.1):

$$\bar{z} = f(z, a, \varepsilon) \approx f_0(z, a) + \varepsilon f_1(z, a) + \dots + \varepsilon^p f_p(z, a), \quad (2.14)$$

where  $z = (z^1, \dots, z^N) \in R^N$  and  $a = (a^1, \dots, a^r)$  is a vector-parameter. It is assumed that

$$f(z, 0, \varepsilon) \approx z.$$

The set  $G_r$  of Transformations 2.14 is called a (local)  $r$ -parameter approximate group of transformations in  $R^N$  if

$$f(f(z, a, \varepsilon), b, \varepsilon) \approx f(z, c, \varepsilon)$$

with the vector-parameter  $c = (c^1, \dots, c^r)$  given by

$$c^\mu = \varphi^\mu(a, b), \quad \mu = 1, \dots, r, \quad (2.15)$$

where  $\varphi^\mu(a, b)$  are smooth functions defined for sufficiently small vector-parameters  $a$  and  $b$ .

Functions 2.15 define a group composition law  $\varphi = (\varphi^1, \dots, \varphi^r)$  and satisfy the following conditions:

$$\varphi(a, 0) = a, \quad \varphi(0, b) = b.$$

### 2.4.2. Infinitesimal generators

Let  $G_r$  be an approximate group. Its infinitesimal generators are the approximate operators (see Section 2.3.1)

$$X_\alpha = \xi_\alpha^i(z, \varepsilon) \frac{\partial}{\partial z^i} \approx (\xi_{\alpha,0}^i(z) + \varepsilon \xi_{\alpha,1}^i(z) + \dots + \varepsilon^p \xi_{\alpha,p}^i(z)) \frac{\partial}{\partial z^i}, \quad (2.16)$$

where  $\alpha = 1, \dots, r$ . The vector fields

$$\xi_{\alpha,v}(z, \varepsilon) = (\xi_{\alpha,v}^1(z, \varepsilon), \dots, \xi_{\alpha,v}^N(z, \varepsilon)), \quad \alpha = 1, \dots, r; \quad v = 1, \dots, p,$$

are determined by the group transformations 2.14 as follows:

$$\xi_{\alpha,v}^i(z, \varepsilon) = \left. \frac{\partial f_v^i(z, a, \varepsilon)}{\partial a^\alpha} \right|_{a=0}, \quad i = 1, \dots, N. \quad (2.17)$$

### 2.4.3. The direct first Lie theorem

Let  $G_r$  be an approximate group of Transformations 2.14 with the composition law 2.15. We set

$$A_{\beta}^{\alpha}(a) = \left. \frac{\partial \varphi^{\alpha}(a, b)}{\partial b^{\beta}} \right|_{b=0}, \quad \alpha, \beta = 1, \dots, r.$$

It follows from the properties of the group composition law  $\varphi(a, b)$  that (see Section 2.4.1):

$$A_{\beta}^{\alpha}(0) = \delta_{\beta}^{\alpha}$$

where  $\delta_{\beta}^{\alpha}$  is the Kronecker symbol. Hence, the matrix  $A = (A_{\beta}^{\alpha}(a))$  has the inverse matrix  $V = (V_{\gamma}^{\beta}(a))$  when  $a$  is sufficiently small. Lie's direct first theorem is valid for the approximate group of Transformations 2.14 in the following modification.

**THEOREM 2.3.** *The functions  $f^i(z, a, \varepsilon)$  satisfy the equations*

$$\frac{\partial f^i}{\partial a^{\alpha}} \approx \xi_{\beta}^i(f, \varepsilon) V_{\alpha}^{\beta}(a), \quad i = 1, \dots, N; \quad \alpha = 1, \dots, r, \quad (2.18)$$

*called the approximate Lie equations. Furthermore, the infinitesimal generators 2.16 span an  $r$ -dimensional approximate Lie algebra.*

### 2.4.4. The inverse first Lie theorem

Consider an arbitrary set of  $r$  linearly independent (to a given precision, see Section 2.3.5) vector fields  $\xi_{\alpha}(z, \varepsilon) = (\xi_{\alpha}^1(z, \varepsilon), \dots, \xi_{\alpha}^N(z, \varepsilon))$ ,  $\alpha = 1, \dots, r$ , and a matrix  $V(a) = (V_{\alpha}^{\beta}(a))$  of the rank  $r$ . Consider, for the given  $\xi_{\alpha}$  and  $V(a)$ , the system of the approximate equations 2.18. Assume that this system is completely integrable (Section 2.1.3). Then the solution of the equations 2.18, under the initial conditions

$$f|_{a=0} \approx z,$$

yields a local  $r$ -parameter approximate group  $G_r$  of Transformations 2.14.

### 2.4.5. Structure constants of an approximate group

Under the assumptions of Section 2.4.4, the coefficients  $V_{\alpha}^{\beta}(a)$  of Equations 2.18 satisfy the Maurer-Cartan equations:

$$\frac{\partial V_{\beta}^{\sigma}}{\partial a^{\alpha}} - \frac{\partial V_{\alpha}^{\sigma}}{\partial a^{\beta}} = c_{\gamma\mu}^{\sigma} V_{\beta}^{\mu} V_{\alpha}^{\gamma}$$

with constant coefficients  $c_{\gamma\mu}^\sigma$ ,  $\gamma, \mu, \sigma = 1, \dots, r$ .

The coefficients  $c_{\gamma\mu}^\sigma$  are termed *the structure constants* of the approximate group  $G_r$  given in Section 2.4.4.

#### 2.4.6. The second Lie theorem

**THEOREM 2.4.** *Consider a given local  $r$ -parameter approximate group  $G_r$ . Let  $c_{\gamma\mu}^\sigma$ ,  $\gamma, \mu, \sigma = 1, \dots, r$ , be its structure constants. Then the infinitesimal generators 2.16 of  $G_r$  are linearly independent (Section 2.3.5) and span an  $r$ -dimensional approximate Lie algebra  $L_r$ . The structure constants of the algebra  $L_r$  (Section 2.3.8) are identical with those of the approximate group  $G_r$ . The approximate Lie algebra  $L_r$  is called the Lie algebra of the approximate group  $G_r$ .*

Conversely, let  $r$  linearly independent approximate operators  $X_\alpha$ ,  $\alpha = 1, \dots, r$ , span an approximate Lie algebra  $L_r$ . Then there exists a local  $r$ -parameter approximate group  $G_r$  such that its Lie algebra is identical to the approximate Lie algebra  $L_r$ .

Given an approximate Lie algebra  $L_r$ , the group  $G_r$  is determined uniquely. One can determine the approximate transformations of the group  $G_r$  by constructing the Lie equations 2.18 or by composing an  $r$ -parameter group from  $r$  one-parameter groups generated by the basic operators of the algebra  $L_r$ . Both ways were previously discussed in this Handbook in the case of classical Lie group theory (see [H2], Sections 1.5.3 and 1.5.5) and can be easily modified to the case of approximate group theory. The second way (the composition of  $G_r$  from one-parameter approximate group transformations) is simple. The first way is more complicated for the practical use. However, it is of fundamental theoretical value. Therefore, for the convenience of the reader, we sketch it in the next section.

#### 2.4.7. Approximate Lie equations for a given approximate Lie algebra

Given an  $r$ -dimensional approximate Lie algebra  $L_r$  with the basic operators 2.16, the problem of determining the completely integrable system of the approximate Lie equations 2.18 reduces to the solution of the Maurer-Cartan equations (Section 2.4.5). The solution of the latter problem is given by the following construction.

Let  $c_{\alpha\beta}^\gamma$ ,  $\alpha, \beta, \gamma = 1, \dots, r$  be the structure constants of the approximate Lie algebra  $L_r$ . Consider the Cauchy problem

$$\frac{d\theta_\alpha^\beta}{dt} = \delta_\alpha^\beta + c_{\nu\mu}^\beta \lambda^\mu \theta_\alpha^\nu, \quad \theta_\alpha^\beta \Big|_{t=0} = 0, \quad (2.19)$$

where  $\delta_\alpha^\beta$  is the Kronecker symbol and  $\lambda^\mu$  is a system of  $r$  parameters. The solution of the problem 2.19 has the form

$$\theta_\alpha^\beta = t h_\alpha^\beta(\lambda^1 t, \dots, \lambda^r t).$$

Define the functions  $V_\alpha^\beta(b)$  as follows:

$$V_\alpha^\beta(b) = h_\alpha^\beta(b^1, \dots, b^r), \quad b^v = \lambda^v t. \quad (2.20)$$

Then the functions  $V_\alpha^\beta(b)$  solve the Maurer-Cartan equations, and hence the system of the approximate Lie equations 2.18 is completely integrable.

#### 2.4.8. Example (the first order of precision)

Consider the approximate Lie algebra  $L_r$  discussed in Section 2.3.9. In this case, the Cauchy problem 2.19 has the following solution:

$$\theta_\alpha^\alpha = t, \quad \alpha = 1, \dots, 5, \quad \theta_1^4 = \frac{1}{2}\lambda^3 t^2, \quad \theta_2^5 = \frac{1}{2}\lambda^3 t^2,$$

and  $\theta_\alpha^\beta = 0$  for other values of  $\alpha$  and  $\beta$ . Formula 2.20 yields:

$$V_\alpha^\alpha = 1, \quad \alpha = 1, \dots, 5; \quad V_1^4 = \frac{1}{2}b^3; \quad V_2^5 = \frac{1}{2}b^3,$$

and  $V_\alpha^\beta = 0$  for other values of  $\alpha$  and  $\beta$ . Thus, by setting  $b^3 = 2a^3$ , one can write the approximate Lie equations 2.18 in the form:

$$\frac{\partial \bar{x}}{\partial a^1} = 1 + \varepsilon a^3, \quad \frac{\partial \bar{x}}{\partial a^2} = \varepsilon \bar{y}, \quad \frac{\partial \bar{x}}{\partial a^3} = \varepsilon \bar{x}, \quad \frac{\partial \bar{x}}{\partial a^4} = \varepsilon, \quad \frac{\partial \bar{x}}{\partial a^5} = 0;$$

$$\frac{\partial \bar{y}}{\partial a^1} = \varepsilon \bar{x}, \quad \frac{\partial \bar{y}}{\partial a^2} = 1 + \varepsilon a^3, \quad \frac{\partial \bar{y}}{\partial a^3} = \varepsilon \bar{y}, \quad \frac{\partial \bar{y}}{\partial a^4} = 0, \quad \frac{\partial \bar{y}}{\partial a^5} = \varepsilon.$$

This system, together with the initial conditions

$$\bar{x}|_{a=0} = x, \quad \bar{y}|_{a=0} = y,$$

yield

$$\bar{x} \approx x + a^1 + \varepsilon \left( a^4 + a^3 x + a^2 y + \frac{1}{2}(a^2)^2 + a^1 a^3 \right),$$

$$\bar{y} \approx y + a^2 + \varepsilon \left( a^5 + a^3 y + a^1 x + \frac{1}{2}(a^1)^2 + a^2 a^3 \right).$$

## 2.5. Equations with a small parameter: Deformations of symmetry Lie algebras

The purpose of this section is to carry over Lie's infinitesimal method (cf. [H2], Sections 1.1.8, 1.2.4, and 1.3.3) to approximate group analysis of differential equations with a small parameter. We sketch the main algorithms for calculating approximate symmetries. A detailed presentation with the proofs of the basic statements is to be found in Baikov, Gazizov, and Ibragimov [1989a].

### 2.5.1. Approximate invariance and the determining equation

Consider a set of smooth  $s$ -dimensional vector-functions

$$F_0(z), F_1(z), \dots, F_q(z)$$

with coordinates

$$F_0^\sigma(z), F_1^\sigma(z), \dots, F_q^\sigma(z), \quad \sigma = 1, \dots, s.$$

It is supposed that  $s \leq N$ .

Let  $G$  be an approximate group of Transformations 2.6 with the given order of approximation  $p \geq q$ . The approximate equation

$$F(z, \varepsilon) \equiv F_0(z) + \varepsilon F_1(z) + \dots + \varepsilon^q F_q(z) = o(\varepsilon^q) \quad (2.21)$$

is said to be approximately invariant with respect to  $G$  if

$$F(f(z, a, \varepsilon), \varepsilon) = o(\varepsilon^q)$$

whenever  $z = (z^1, \dots, z^N)$  satisfies Equation 2.21.

**THEOREM 2.5.** *Let*

$$\text{rank} \left\| \frac{\partial F_0^\sigma(z)}{\partial z^i} \right\|_{F_0(z)=0} = s.$$

*Let  $X$  be the Generator 2.8 of the group  $G$ . Then Equation 2.21 is approximately invariant under the approximate group  $G$  if and only if*

$$XF(z, \varepsilon) \Big|_{(2.21)} = o(\varepsilon^p). \quad (2.22)$$

Equation 2.22 is called *the determining equation* for approximate symmetries. If the determining equation 2.22 is satisfied we also say that *Equation 2.21 admits the approximate operator  $X$* .

### 2.5.2. Deformations of a symmetry Lie algebra. Stable symmetries

Theorem 2.5 yields the following simple result which is, however, of fundamental importance.

**THEOREM 2.6.** *Let the equation 2.21 be approximately invariant under the approximate group with the generator 2.9:*

$$X = \xi^i(z, \varepsilon) \frac{\partial}{\partial z^i} \equiv \left( \xi_0^i(z) + \varepsilon \xi_1^i(z) + \cdots + \varepsilon^p \xi_p^i(z) \right) \frac{\partial}{\partial z^i},$$

such that  $\xi_0(z) = (\xi_0^1(z), \dots, \xi_0^N(z)) \neq 0$ . Then the (exact) operator

$$X^0 = \xi_0^i(z) \frac{\partial}{\partial z^i} \tag{2.23}$$

is the generator of an exact symmetry group for the equation

$$F_0(z) = 0. \tag{2.24}$$

In what follows, Equation 2.24 is treated as an *unperturbed equation*, and Equation 2.21 is termed a *perturbed equation*. Under the conditions of Theorem 2.6, the exact symmetry generator  $X^0$  is called a *stable symmetry* of the unperturbed equation 2.24. The corresponding approximate symmetry generator  $X$  for the perturbed equation 2.21 is called a *deformation* (of order  $p$ ) of the operator  $X^0$  caused by the perturbation of order  $q$ , viz.

$$\varepsilon F_1(z) + \cdots + \varepsilon^q F_q(z).$$

The notions of a stable symmetry and of its deformations apply to any symmetry Lie algebra of Equation 2.24. In particular, it may happen that the most general symmetry Lie algebra of Equation 2.24 is stable. In this particular case we say that the perturbed equation 2.21 *inherits the symmetries of the unperturbed equation*.

### 2.5.3. First-order deformations

Here, we simplify the equations of Section 2.5.1 by letting  $p = q = 1$ . Then Equation 2.21 and an approximate group generator 2.9 become

$$F_0(z) + \varepsilon F_1(z) \approx 0$$

and

$$X = X^0 + \varepsilon X^1 \equiv \xi_0^i(z) \frac{\partial}{\partial z^i} + \varepsilon \xi_1^i(z) \frac{\partial}{\partial z^i},$$

respectively. The determining equation 2.22 for approximate symmetries reduces to the following:

$$(X^0 + \varepsilon X^1)(F_0(z) + \varepsilon F_1(z)) \Big|_{F_0(z) + \varepsilon F_1(z) = 0} = o(\varepsilon).$$

The determining equation can also be written, by using undeterminate coefficients, in the form similar to the case of exact Lie symmetries (cf. [H2], Section 1.3.3, Equation 1.50'). This form is given by the following theorem (see Baikov, Gazizov, and Ibragimov [1989a], Equations 4.23 to 4.25). For the sake of brevity, we limit here the discussion to scalar equations. That is, we let  $s = 1$ .

**THEOREM 2.7.** *In the first order of precision, the determining equation for approximate symmetries can be written as follows:*

$$X^0 F_0(z) = \lambda(z) F_0(z), \quad (2.25)$$

$$X^1 F_0(z) + X^0 F_1(z) = \lambda(z) F_1(z). \quad (2.26)$$

Here the factor  $\lambda(z)$  is determined by Equation 2.25 and afterwards substituted into Equation 2.26 where Equation 2.26 itself must hold for the solutions  $z$  of the unperturbed equation 2.24, viz.  $F_0(z) = 0$ .

### 2.5.4. Algorithm for calculating the first-order approximate symmetries

Theorems 21 and 22 provide a simple and convenient algorithm for calculating (both "by hand" and by using symbolic software) the first-order approximate symmetries of equations with a small parameter. The algorithm consists of the following three steps.

*1st step.* Find the exact symmetry generators  $X^0$  of the unperturbed Equation 2.24, e.g., by solving the determining equation for exact symmetries:

$$X^0 F_0(z) \Big|_{F_0(z)=0} = 0.$$

*image  
not  
available*

derivatives involved in Equation 2.27, the operator  $X$  has the form:<sup>2</sup>

$$X = (\xi_0^0 + \varepsilon \xi_1^0) \frac{\partial}{\partial t} + (\xi_0^1 + \varepsilon \xi_1^1) \frac{\partial}{\partial x} + (\eta_0 + \varepsilon \eta_1) \frac{\partial}{\partial u} + (\zeta_{0,0} + \varepsilon \zeta_{1,0}) \frac{\partial}{\partial u_t} \\ + (\zeta_{0,1} + \varepsilon \zeta_{1,1}) \frac{\partial}{\partial u_x} + (\zeta_{0,00} + \varepsilon \zeta_{1,00}) \frac{\partial}{\partial u_{tt}} + (\zeta_{0,11} + \varepsilon \zeta_{1,11}) \frac{\partial}{\partial u_{xx}}.$$

Here, the prolongation formulas are

$$\zeta_{v,0} = D_t(\eta_v) - u_t D_t(\xi_v^0) - u_x D_t(\xi_v^1),$$

$$\zeta_{v,1} = D_x(\eta_v) - u_t D_x(\xi_v^0) - u_x D_x(\xi_v^1),$$

$$\zeta_{v,00} = D_t(\zeta_{v,0}) - u_{tt} D_t(\xi_v^0) - u_{tx} D_t(\xi_v^1),$$

$$\zeta_{v,11} = D_x(\zeta_{v,1}) - u_{tx} D_x(\xi_v^0) - u_{xx} D_x(\xi_v^1),$$

where  $v = 0, 1$ . The operators  $D_t$  and  $D_x$  are the total differentiations 1.2 (see Chapter 1) with respect to the independent variables  $t$  and  $x$ , viz.

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + \dots, \quad D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + \dots$$

The algorithm of Section 2.5.4 requires the calculation of exact symmetries for the unperturbed equation

$$u_{tt} = (u^\sigma u_x)_x, \quad \sigma \neq 0. \quad (2.28)$$

We will use the available result of the group classification of nonlinear wave equations due to Ames, Lohner, and Adams [1981] (see also [H1], Section 12.4.1). This classification singles out three types of Equations 2.28 corresponding to the following values of  $\sigma$ : (i)  $\sigma \neq 0$  is an arbitrary constant, (ii)  $\sigma = -4/3$ , (iii)  $\sigma = -4$ . Let's implement the algorithm in these three distinct cases.

<sup>2</sup>Editor's note: In contemporary group analysis based on the Lie, Lie-Bäcklund, and approximate group theories, infinitesimal generators act in the *universal space*  $\mathcal{A}$  of differential functions (see Chapter 1). Given an operator

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}, \quad \xi^i, \eta^\alpha \in \mathcal{A},$$

its extension to derivatives (of any order) of  $u$  is uniquely determined by the prolongation formulas. Therefore, there is no ambiguity if one uses (as S. Lie did) the same symbol  $X$  for the prolonged action of this operator. It's no advantage introducing special symbols to designate prolonged groups and their infinitesimal generators.

**I.  $\sigma \neq 0$  is an arbitrary constant.**

*1st step.* The most general Lie symmetry generator for Equation 2.28 is

$$X^0 = (C_1 + C_3 t) \frac{\partial}{\partial t} + (C_2 + C_3 x + C_4 x) \frac{\partial}{\partial x} + C_4 \frac{2u}{\sigma} \frac{\partial}{\partial u}$$

where  $C_1, \dots, C_4$  are arbitrary constants. Hence, the unperturbed equation admits a 4-dimensional Lie algebra  $L_4$ .

*2nd step.* Using this generator  $X^0$ , one readily finds the auxiliary function

$$H = C_3 u_t.$$

*3rd step.* Thus, we arrive at the following determining equation for deformations:

$$X^1 \left( u_{tt} - u^\sigma u_{xx} + \sigma u^{\sigma-1} u_x^2 \right) \Big|_{(2.28)} + C_3 u_t = 0,$$

where  $X^1$  is the operator

$$X^1 = \xi_1^0 \frac{\partial}{\partial t} + \xi_1^1 \frac{\partial}{\partial x} + \eta_1 \frac{\partial}{\partial u},$$

extended to the derivatives of  $u$  involved in Equation 2.28. To solve this equation, we apply the same approach as in the case of determining equations for Lie symmetries (cf. [H2], Section 1.3.4). Namely, we isolate, in the left-hand side of our determining equation, the terms containing  $u_{tx}$ ,  $u_{xx}$ ,  $u_t$ ,  $u_x$ . As a result, we arrive at a polynomial in the variables  $u_{tx}$ ,  $u_{xx}$ ,  $u_t$ ,  $u_x$ . Then we set the coefficients of this polynomial equal to zero. It follows:

$$\xi_1^0 = \xi_1^0(t), \quad \xi_1^1 = \xi_1^1(x), \quad \eta_1 = \frac{2u}{\sigma} ((\xi_1^1)_x - (\xi_1^0)_t),$$

$$\left( \sigma + \frac{4}{3} \right) (\xi_1^1)_{xx} = 0, \tag{2.29}$$

$$\left( \frac{4}{\sigma} + 1 \right) (\xi_1^0)_{tt} = C_3, \quad (\xi_1^0)_{ttt} = 0, \quad (\xi_1^1)_{xxx} = 0.$$

The general solution of Equations 2.29 is

$$\xi_1^0 = \frac{\sigma}{2(\sigma + 4)} C_3 t^2 + A_1 + A_3 t, \quad \xi_1^1 = A_2 + A_3 x + A_4 x,$$

$$\eta_1 = \frac{2u}{\sigma} \left( A_4 - \frac{\sigma}{\sigma + 4} C_3 t \right).$$

Thus, if  $\sigma \neq 0$  is an arbitrary constant, Equation 2.27 has the following first-order approximate symmetry generators:

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, & X_2 &= \frac{\partial}{\partial x}, \\ X_3 &= \left(t + \frac{\varepsilon \sigma t^2}{2(\sigma + 4)}\right) \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} - \frac{2\varepsilon t u}{\sigma + 4} \frac{\partial}{\partial u}, & X_4 &= x \frac{\partial}{\partial x} + \frac{2u}{\sigma} \frac{\partial}{\partial u}, \\ X_5 &= \varepsilon \frac{\partial}{\partial t}, & X_6 &= \varepsilon \frac{\partial}{\partial x}, \\ X_7 &= \varepsilon \left(t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x}\right), & X_8 &= \varepsilon \left(x \frac{\partial}{\partial x} + \frac{2u}{\sigma} \frac{\partial}{\partial u}\right). \end{aligned}$$

These operators span an approximate 8-dimensional Lie algebra  $L_8$ . Its essential operators (see Section 2.3.7) are  $X_1, X_2, X_3, X_4$ . Further, the approximate Lie algebra  $L_8$  is a first-order deformation of the most general symmetry Lie algebra  $L_4$  for Equation 2.28. That is, the algebra  $L_4$  is stable. Hence, according to Section 2.5.2, the perturbed equation 2.27 inherits the symmetries of the unperturbed equation 2.28 with arbitrary  $\sigma$ .

**II.  $\sigma = -4/3$ .**

*1st step.* In this case, the most general Lie symmetry generator for Equation 2.28 is

$$X^0 = (C_1 + C_3 t) \frac{\partial}{\partial t} + (C_2 + C_3 x + C_4 x + C_5 x^2) \frac{\partial}{\partial x} - \left(\frac{3}{2} C_4 u + 3 C_5 x u\right) \frac{\partial}{\partial u}$$

where  $C_1, \dots, C_5$  are arbitrary constants. Hence, the unperturbed equation admits a 5-dimensional Lie algebra  $L_5$ .

*2nd step.* Here, the auxiliary function has the form  $H = C_3 u_t$ .

*3rd step.* The determining equation for deformations is given by the System 2.29 with  $\sigma = -4/3$ . It follows:

$$\xi_1^0 = -\frac{C_3}{4} t^2 + A_1 + A_3 t, \quad \xi_1^1 = A_2 + A_3 x + A_4 x + A_5 x^2,$$

$$\eta_1 = -\frac{3}{4} C_3 t u - 3 A_5 x u - \frac{3}{2} A_4 u.$$

Thus, Equation 2.27 with  $\sigma = -4/3$  admits a 10-dimensional approximate Lie algebra  $L_{10}$ . It is a first-order deformation of the symmetry Lie algebra  $L_5$  for the corresponding unperturbed equation 2.28. The essential operators of  $L_{10}$  are

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \left(t - \frac{\varepsilon}{4} t^2\right) \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} - \frac{3}{4} \varepsilon t u \frac{\partial}{\partial u},$$

$$X_4 = x \frac{\partial}{\partial x} - \frac{3}{2} u \frac{\partial}{\partial u}, \quad X_5 = x^2 \frac{\partial}{\partial x} - 3xu \frac{\partial}{\partial u}.$$

In this case, the maximal symmetry algebra  $L_5$  is also stable, i.e., the perturbed equation inherits the symmetries of the unperturbed equation.

**III.**  $\sigma = -4$ .

*1st step.* The most general Lie symmetry generator for Equation 2.28 is

$$X^0 = (C_1 + C_3t + C_5t^2) \frac{\partial}{\partial t} + (C_2 + C_3x + C_4x) \frac{\partial}{\partial x} + \left(-\frac{C_4}{2}u + C_5tu\right) \frac{\partial}{\partial u}$$

where  $C_1, \dots, C_5$  are arbitrary constants. Hence, the unperturbed equation admits a 5-dimensional Lie algebra  $L_5$ .

*2nd step.* The auxiliary function has the form  $H = C_3u_t + 2C_5tu_t + C_5u$ .

*3rd step.* The determining equation for deformations reduces to the following system:

$$\xi_1^0 = \xi_1^0(t), \quad \xi_1^1 = \xi_1^1(x), \quad \eta_1 = -\frac{u}{2} \left( (\xi_1^1)_x - (\xi_1^0)_t \right),$$

$$\frac{C_3}{2} + C_5t = 0, \quad (\xi_1^1)_{xx} = 0, \quad (\xi_1^0)_{ttt} = -C_5.$$

The general solution of this system is

$$\xi_1^0 = A_1 + A_3t + A_5t^2, \quad \xi_1^1 = A_2 + A_3x + A_4x,$$

$$\eta_1 = -\frac{u}{2}(A_4 - 2A_5t), \quad C_3 = C_5 = 0.$$

It follows that the operators

$$X_3^0 = t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x}, \quad X_5^0 = t^2 \frac{\partial}{\partial t} + tu \frac{\partial}{\partial u}$$

are not stable (see Section 2.5.2).

Thus, Equation 2.27 with  $\sigma = -4$  admits an 8-dimensional approximate Lie algebra  $L_8$ . Its essential operators are

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \varepsilon \left( t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} \right), \quad X_4 = x \frac{\partial}{\partial x} - \frac{u}{2} \frac{\partial}{\partial u},$$

$$X_5 = \varepsilon \left( t^2 \frac{\partial}{\partial t} + tu \frac{\partial}{\partial u} \right).$$

In this case, the maximal symmetry algebra  $L_5$  is not stable. Hence, the perturbed equation does not inherit the symmetries of the unperturbed equation.

differential variables  $y$  and  $f$ , the operator  $Y$  has the form:

$$Y = (\xi_0 + \varepsilon \xi_1) \frac{\partial}{\partial x} + (\eta_0 + \varepsilon \eta_1) \frac{\partial}{\partial y} + \phi \frac{\partial}{\partial f} + \zeta_1 \frac{\partial}{\partial y'} + \zeta_{11} \frac{\partial}{\partial y''} + \omega_1 \frac{\partial}{\partial f_x}. \quad (2.32)$$

Here, the coefficients  $\zeta_1$  and  $\zeta_{11}$  are given by the prolongation formulas of Section 2.5.5, and the coefficient  $\omega_1$  is obtained by the similar formula applied to the differential function  $f$  with the independent variables  $x, y$ , viz.

$$\omega_1 = \tilde{D}_x(\phi) - f_x \tilde{D}_x(\xi_0 + \varepsilon \xi_1) - f_y \tilde{D}_x(\eta_0 + \varepsilon \eta_1),$$

where (cf.  $D_x$  of Section 2.5.5)

$$\tilde{D}_x = \frac{\partial}{\partial x} + f_x \frac{\partial}{\partial f} + \dots$$

The infinitesimal approximate invariance criterion of System 2.31 has the form:

$$Y(y'' - \varepsilon f)|_{(2.31)} = o(\varepsilon), \quad Y(\varepsilon f_x)|_{(2.31)} = o(\varepsilon), \quad (2.33)$$

where  $Y$  is the operator 2.32.

In the zero order of precision, the first equation 2.33 yields:

$$Y(y'')|_{y''=0} = 0.$$

This is the determining equation for exact symmetries of the unperturbed equation  $y'' = 0$ . Hence (see, e.g., [H1], Section 8.4),

$$\xi_0 = (C_1 x + C_2) y + C_3 x^2 + C_4 x + C_5,$$

$$\eta_0 = C_1 y^2 + C_3 x y + C_6 y + C_7 x + C_8,$$

where  $C_1, \dots, C_8$  are arbitrary constants.

Further, in the first order of precision, the second equation 2.33 yields:

$$\phi_x = 0, \quad (\eta_0)_x = 0.$$

Then the first equation 2.33 reduces to the following system:

$$(\xi_1)_{yy} = 0, \quad (\eta_1)_{yy} - 2(\xi_1)_{xy} = 0, \quad 2(\eta_1)_{xy} - (\xi_1)_{xx} - 3(\xi_0)_y f = 0,$$

$$\phi = (\eta_1)_{xx} + ((\eta_0)_y - 2(\xi_0)_x) f.$$

Since the coefficients  $\xi$  and  $\eta$  do not depend on the differential variable  $f$ , the above equations yield:

$$C_1 = 0, \quad C_2 = 0, \quad C_3 = 0, \quad C_7 = 0,$$

$$\xi_1 = (A_1x + A_2)y + 2B_1x^3 + A_3x^2 + A_4x + A_5,$$

$$\eta_1 = A_1y^2 + 3B_1x^2y + A_3xy + A_6y + B_2x^2 + A_7x + A_8,$$

$$\phi = 6B_1y + 2B_2 + (C_6 - 2C_4)f.$$

Thus, Equation 2.30 admits a 14-dimensional approximate Lie algebra  $L_{14}$  of infinitesimal generators of approximate equivalence transformations. The algebra  $L_{14}$  is spanned by

$$Y_1 = x \frac{\partial}{\partial x} - 2f \frac{\partial}{\partial f}, \quad Y_2 = \frac{\partial}{\partial x}, \quad Y_3 = y \frac{\partial}{\partial y} + f \frac{\partial}{\partial f},$$

$$Y_4 = \frac{\partial}{\partial y}, \quad Y_5 = 2\epsilon x^3 \frac{\partial}{\partial x} + 3\epsilon x^2 y \frac{\partial}{\partial y} + 6y \frac{\partial}{\partial f}, \quad Y_6 = \epsilon x^2 \frac{\partial}{\partial y} + 2 \frac{\partial}{\partial f},$$

$$Y_7 = \epsilon \left( xy \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y} \right), \quad Y_8 = \epsilon y \frac{\partial}{\partial x}, \quad Y_9 = \epsilon \left( x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y} \right),$$

$$Y_{10} = \epsilon x \frac{\partial}{\partial x}, \quad Y_{11} = \epsilon \frac{\partial}{\partial x}, \quad Y_{12} = \epsilon y \frac{\partial}{\partial y}, \quad Y_{13} = \epsilon x \frac{\partial}{\partial y}, \quad Y_{14} = \epsilon \frac{\partial}{\partial y}.$$

The corresponding 14-parameter approximate equivalence transformation group is given by:

$$\tilde{x} = a_1x + a_2 + \epsilon(2a_1a_5x^3 + a_1a_7xy + a_8y + a_1a_9x^2 + a_{10}x + a_{11}),$$

$$\tilde{y} = a_3y + a_4 + \epsilon(3a_3a_5x^2y + a_6x^2 + a_3a_7y^2 + a_3a_9xy + a_{12}y + a_{13}x + a_{14}),$$

$$\tilde{f} = \frac{a_3}{a_1^2} f + 6 \frac{a_3a_5}{a_1^2} y + 2 \frac{a_6}{a_1^2}.$$

## 2.6. Approximate conservation laws

### 2.6.1. Adaptation of the Noether theorem

Here, the discussion is restricted to the case of Lagrangians  $L(x, u, u_{(1)}, \varepsilon)$  depending on an independent variable  $x = (x^1, \dots, x^n)$ , a dependent variable  $u = (u^1, \dots, u^m)$ , and the first-order derivatives  $u_{(1)} = \{u_i^\alpha\}$  of  $u$  with respect to  $x$ . Thus,  $L \in \mathcal{A}$  is a differential function of the first order (see Section 1.1.1) depending also on a small parameter  $\varepsilon$ . Further restriction is that approximate groups are limited by point transformations. The general case involving higher-order Lagrangians and approximate Lie-Bäcklund transformation groups can be treated in a similar way.

**THEOREM 2.8.** (Cf. [H1], Section 6.2.) Consider an approximate Euler-Lagrange equation, viz.

$$\frac{\delta L}{\delta u^\alpha} \equiv \frac{\partial L}{\partial u^\alpha} - D_i \left( \frac{\partial L}{\partial u_i^\alpha} \right) = o(\varepsilon^q). \quad (2.34)$$

Let Equation 2.34 be invariant under the approximate (of an arbitrary order  $p$ ) group of point transformations with the approximate generator

$$X \approx \xi^i(x, u, \varepsilon) \frac{\partial}{\partial x^i} + \eta^\alpha(x, u, \varepsilon) \frac{\partial}{\partial u^\alpha},$$

where (cf. Operator 2.9)

$$\xi^i(x, u, \varepsilon) = \xi_0^i(x, u) + \varepsilon \xi_1^i(x, u) + \dots + \varepsilon^p \xi_p^i(x, u),$$

$$\eta^\alpha(x, u, \varepsilon) = \eta_0^\alpha(x, u) + \varepsilon \eta_1^\alpha(x, u) + \dots + \varepsilon^p \eta_p^\alpha(x, u).$$

Let  $X$  be an approximate Noether symmetry, i.e.,

$$XL + LD_i(\xi^i) = D_i(B^i) + o(\varepsilon^p) \quad (2.35)$$

with  $B^i \in \mathcal{A}$ ,  $i = 1, \dots, n$ . Then the differential functions

$$C^i = L\xi^i + (\eta^\alpha - \xi^j u_j^\alpha) \frac{\partial L}{\partial u_i^\alpha} - B^i + o(\varepsilon^p) \quad (2.36)$$

satisfy the approximate conservation law for Equation 2.34:

$$D_t(C^i) \Big|_{(2.34)} = o(\varepsilon^p). \quad (2.37)$$

### 2.6.2. An application (the first order of precision)

The equation

$$u_{tt} + \varepsilon u_t - u_{xx} - u_{yy} = 0$$

has the Lagrangian

$$L = \frac{1}{2} e^{\varepsilon t} (u_t^2 - u_x^2 - u_y^2)$$

and admits the approximate operator (see Section 9.2.2.1)

$$X = (t^2 + x^2 + y^2) \frac{\partial}{\partial t} + 2tx \frac{\partial}{\partial x} + 2ty \frac{\partial}{\partial y} - \left( t + \frac{\varepsilon}{2} (t^2 + x^2 + y^2) \right) u \frac{\partial}{\partial u}.$$

The condition 2.35 holds with the functions

$$B^1 \approx -(1 + \varepsilon t)u^2/2, \quad B^2 \approx \varepsilon x u^2/2, \quad B^3 \approx \varepsilon y u^2/2.$$

Formula 2.36 yields the following approximate conservation law:

$$\begin{aligned} D_t \left( e^{\varepsilon t} \left[ -\frac{s^2}{2} (u_t^2 + u_x^2 + u_y^2) - u_t \left( tu + \frac{\varepsilon}{2} s^2 u + 2txu_x + 2tyu_y \right) \right] + \frac{1 + 2\varepsilon t}{2} u^2 \right) \\ + D_x \left( e^{\varepsilon t} \left[ tx(u_t^2 + u_x^2 - u_y^2) + u_x \left( tu + \frac{\varepsilon}{2} s^2 u + s^2 u_t - \frac{\varepsilon}{2} x u^2 + 2tyu_y \right) \right] \right) \\ + D_y \left( e^{\varepsilon t} \left[ ty(u_t^2 - u_x^2 + u_y^2) + u_y \left( tu + \frac{\varepsilon}{2} s^2 u + s^2 u_t - \frac{\varepsilon}{2} y u^2 + 2txu_x \right) \right] \right) = o(\varepsilon), \end{aligned}$$

where  $s^2 = t^2 + x^2 + y^2$ .

---

## 2.7. Approximately invariant solutions (the first order of precision)

Cf. [H1], Section 4.1 or [H2], Section 1.5.11.

### 2.7.1. An illustrative example

The equation

$$u_{tt} + \varepsilon u_t = (u^\sigma u_x)_x, \quad \sigma \neq -4, \quad (2.38)$$

admits the approximate operator (see Section 2.5.5)

$$X = X^0 + \varepsilon X^1 \equiv t \frac{\partial}{\partial t} - \frac{2u}{\sigma} \frac{\partial}{\partial u} + \varepsilon \left( \frac{\sigma}{2\sigma + 8} t^2 \frac{\partial}{\partial t} - \frac{2}{\sigma + 4} t u \frac{\partial}{\partial u} \right). \quad (2.39)$$

It is sufficient, for purposes of illustration, to consider *regular invariant solutions* (see [H2], Section 1.5.11) written via invariants. Approximate invariants

$$J(t, x, u, \varepsilon) \approx J_0(t, x, u) + \varepsilon J_1(t, x, u)$$

for the operator 2.39 are determined by the equation

$$XJ = o(\varepsilon),$$

or equivalently,

$$X^0 J_0 + \varepsilon (X^1 J_0 + X^0 J_1) = 0.$$

This equation splits into the system:

$$X^0 J_0 = 0, \quad X^0 J_1 = -X^1 J_0.$$

It follows that the operator  $X$  has two functionally independent approximate invariants given by

$$J^1 \approx x + \varepsilon \alpha(x, ut^{2/\sigma}), \quad J^2 \approx ut^{2/\sigma} + \varepsilon \left[ \frac{ut^{2/\sigma+1}}{\sigma + 4} + \beta(x, ut^{2/\sigma}) \right]$$

with arbitrary functions  $\alpha$  and  $\beta$ .

In the simple case when  $\alpha = \beta = 0$ , an approximately invariant solution given by the equation  $J^2 \approx \varphi(J^1)$  has the form

$$u \approx t^{-2/\sigma} \left( 1 - \frac{\varepsilon t}{\sigma + 4} \right) \varphi(x). \quad (2.40)$$

The substitution of the function 2.40 into Equation 2.38 yields

$$(\varphi^\sigma \varphi')' = \frac{2\sigma + 4}{\sigma^2} \varphi. \quad (2.41)$$

approximate solution 2.44, depending on two arbitrary constants. In the particular case when

$$\psi_2 = -\frac{\psi_1}{\sigma + 4},$$

the system 2.45 – 2.46 provides a solution of the type 2.44 approximately invariant with respect to the operator 2.39. However, in general, the solutions of the form 2.44 can not be obtained as approximately invariant solutions by using the approximate symmetries given in Section 2.5.5. This is due to the fact that, according to Section 2.5.5, the symmetry

$$X^0 = t \frac{\partial}{\partial t} - \frac{2u}{\sigma} \frac{\partial}{\partial u}$$

is not stable for Equation 2.42 with  $\sigma = -4$ .

Note that Operator 2.43 generates a group of *exact* equivalence transformations. However, the same approach applies to approximate equivalence transformations.

## 2.8. Formal symmetries

The generalization of the previous considerations to infinite-order approximate symmetries leads to what is called a *formal symmetry* and a *formal Bäcklund transformation* (Baikov, Gazizov, and Ibragimov [1987b], [1988d], [1989b]). Here, the approach is applied to evolution equations of the form  $u_t = h(u)u_1 + \varepsilon H(t, x, u, u_1, \dots, u_n)$ . In particular, a new viewpoint to Lie-Bäcklund symmetries offered by formal Bäcklund transformations is sketched and illustrated by the Korteweg-de Vries equation.

The notation is taken from Chapter 1, Sections 1.1.1 and 1.4.3. Namely:  $t$  and  $x$  are independent variables,  $u$  is a differential variable with successive derivatives (with respect to  $x$ )  $u_1 = u_x$ ,  $u_2 = u_{xx}$ ,  $\dots$ , so that  $u_{i+1} = D(u_i)$ ,  $u_0 = u$ , where

$$D = \frac{\partial}{\partial x} + u_1 \frac{\partial}{\partial u} + \dots$$

is the total differentiation with respect to  $x$ .  $\mathcal{A}$  is the space of differential functions depending on the variables  $t, x, u, u_1, \dots$ . In addition, the following abbreviations are used for derivatives of differential functions  $\eta$ :

$$\eta_t = \frac{\partial \eta}{\partial t}, \quad \eta_x = \frac{\partial \eta}{\partial x}, \quad \eta_i = \frac{\partial \eta}{\partial u_i}, \quad \eta_* = \sum_{i \geq 0} \eta_i D^i.$$

### 2.8.1. Formal symmetries for the equation $u_t = h(u)u_1 + \varepsilon H$

All Lie-Bäcklund symmetries of the first-order evolutionary equation

$$u_t = h(u)u_1 \quad (2.47)$$

with an arbitrary function  $h(u)$  are stable under the perturbation  $\varepsilon H$  with any differential function  $H \in \mathcal{A}$ . That is, a perturbed equation of the form

$$u_t = h(u)u_1 + \varepsilon H, \quad H \in \mathcal{A}, \quad (2.48)$$

inherits the symmetries of the unperturbed equation 2.47 as defined in Section 2.5.2. Moreover, the following statement about the infinite-order stability is valid (for the proof, see, e.g., Baikov, Gazizov, and Ibragimov [1989a], Section 6.1).

**THEOREM 2.9.** *Given a canonical Lie-Bäcklund operator*

$$X^0 = \eta^0 \frac{\partial}{\partial u} + \dots, \quad \eta^0 \in \mathcal{A},$$

*admitted by the unperturbed equation 2.47, the perturbed equation 2.48 admits an infinite-order deformation*

$$X = \eta \frac{\partial}{\partial u} + \dots,$$

*where  $\eta$  is an element of the space  $[[\mathcal{A}]]$  of formal power series (see Chapter 1, Section 1.2.1), i.e., it has the form*

$$\eta = \sum_{v=0}^{\infty} \varepsilon^v \eta^v, \quad \eta^v \in \mathcal{A}. \quad (2.49)$$

In this theorem, the coefficients  $\eta^0$  of Lie-Bäcklund symmetries  $X^0$  of Equation 2.47 are determined by the equation

$$\eta_t^0 - h(u)\eta_x^0 + \sum_{i \geq 1} [D^i(hu_1) - hu_{i+1}]\eta_i^0 - h'(u)u_1\eta^0 = 0. \quad (2.50)$$

By an inspection the general solution of Equation 2.50 may be verified to be

$$\eta^0 = u_1 \alpha \left( h(u), x + t h(u), t + \frac{1}{h'(u)u_1}, \dots \right) \quad (2.51)$$

with an arbitrary function  $\alpha \in \mathcal{A}$  (see [H1], Section 11.3). Given  $\eta^0$ , the coefficients  $\eta^\nu$ ,  $\nu \geq 1$  of the corresponding power series 2.49 are determined recursively by the following infinite system of linear first-order partial differential equations:

$$\begin{aligned} \eta_t^\nu - h(u)\eta_x^\nu + \sum_{i \geq 1} [D^i(hu_1) - hu_{i+1}]\eta_i^\nu - h'(u)u_1\eta^\nu \\ = \sum_{i \geq 1} [D^i(\eta^{v-1})H_i - \eta_i^{v-1}D^i(H)], \quad \nu = 1, \dots \end{aligned} \quad (2.52)$$

Furthermore, if the solution  $\eta^0 = \eta^0(t, x, u, \dots, u_k)$  of Equation 2.50 and the perturbation  $H = H(t, x, u, \dots, u_n)$  are differential functions of the orders  $k$  and  $n$ , respectively, then the system 2.52 is solved by a sequence of differential functions  $\eta^\nu$  of the orders  $k_\nu = \nu(n-1) + k$ .

An infinite-order deformation  $X$  determined by the formal power series 2.49 is called a *formal symmetry* for Equation 2.48.

### 2.8.2. Lie-Bäcklund via formal symmetries

It is clear that if one truncates arbitrarily the series 2.49 and takes a finite sum

$$\eta = \sum_{\nu=0}^p \varepsilon^\nu \eta^\nu,$$

one obtains an approximate symmetry (up to the truncation error  $o(\varepsilon^p)$ ) for Equation 2.48.

The situation is more interesting when the series 2.49 breaks off for a certain function  $H$  and certain symmetry  $X^0$  of the unperturbed equation 2.47. If this occurs, the corresponding deformation  $X$  has a finite order and provides an *exact Lie-Bäcklund symmetry* for Equation 2.48 with an arbitrary constant  $\varepsilon$ , e.g., with  $\varepsilon = 1$ .

**Example.** Consider a nonlinear equation 2.47:

$$u_t = h(u)u_1, \quad h'(u) \neq 0.$$

For this equation, let us take the simplest Lie-Bäcklund operator given by Formula 2.51, viz.

$$X^0 = \varphi(u)u_1 \frac{\partial}{\partial u}.$$

Let us specify the perturbed equation 2.48 by letting  $H = u_3$ :

$$u_t = h(u)u_1 + \varepsilon u_3.$$

In this particular case, one can prove by inspecting the determining equations 2.52 that for an arbitrary polynomial function  $\varphi(u)$ , the series 2.49 breaks off if and only if  $h''' = 0$ , i.e., if

$$h(u) = au^2 + bu + c, \quad a, b, c = \text{const.}$$

The result of this example agrees with the well-known fact of the existence of Lie-Bäcklund symmetries for the modified Korteweg-de Vries equation

$$u_t = (au^2 + bu + c)u_1 + u_3.$$

Moreover, the new approach leads to a further understanding of the nature of Lie-Bäcklund symmetries. An algorithm for the calculation of exact symmetries provided by this approach is direct and simple. Namely, one takes any polynomial  $\varphi(u)$  and solves Equations 2.52 recursively beginning with  $\eta^0 = \varphi(u)u_1$ . As a result, one obtains an exact Lie-Bäcklund symmetry of order  $2n + 1$  provided that  $\varphi(u)$  is a polynomial of degree  $n$ . Compare Section 9.4.3 of Chapter 9. For a more detailed discussion see Baikov, Gazizov, and Ibragimov [1989a], Section 6.

### 2.8.3. Formal Bäcklund transformations

Equation 2.48 with an arbitrary function  $H(t, x, u, u_1, \dots) \in \mathcal{A}$  can be mapped into the first-order equation 2.47:

$$v_t = h(v)v_1 \tag{2.53}$$

by a substitution  $u \mapsto v$  given by a formal power series of the form

$$v = u + \sum_{i \geq 1} \varepsilon^i \Phi^i(t, x, u, u_1, \dots), \quad \Phi^i \in \mathcal{A}. \tag{2.54}$$

The coefficients  $\Phi^i$  of the transformation 2.54 are determined recursively by the following infinite system of linear first-order partial differential equations:

$$\Phi_t^1 - h(u)\Phi_x^1 + \sum_{\alpha \geq 1} (D^\alpha(hu_1) - hu_{\alpha+1})\Phi_\alpha^1 - h'(u)u_1\Phi^1 = -H, \tag{2.55}$$

$$\Phi_t^i - h(u)\Phi_x^i + \sum_{\alpha \geq 1} (D^\alpha(hu_1) - hu_{\alpha+1})\Phi_\alpha^i - h'(u)u_1\Phi^i$$

$$= - \sum_{\alpha \geq 0} \Phi_\alpha^{i-1} D^\alpha(H) + u_1 \sum_{k=2}^i \frac{1}{k!} h^{(k)}(u) \sum_{i_1 + \dots + i_k = i} \Phi^{i_1} \dots \Phi^{i_k}$$

$$+ \sum_{j+l=i} D(\Phi^j) \left( \sum_{k=1}^l \frac{1}{k!} h^{(k)}(u) \sum_{i_1+\dots+i_k=l} \Phi^{i_1} \dots \Phi^{i_k} \right), \quad i \geq 2. \quad (2.56)$$

Provided that  $H = H(t, x, u, \dots, u_n)$  is a differential function of the order  $n$ , the system 2.55 – 2.56 can be solved by differential functions  $\Phi^k$  of the orders  $nk$ .

The transformation 2.54 is called a *formal Bäcklund transformation* relating Equations 2.48 and 2.53.

The point transformation

$$y = h(v), \quad w = x + t h(v)$$

reduces Equation 2.53 to the linear equation

$$w_t = 0 \quad (2.57)$$

for the function  $w = w(t, y)$ . Hence, any equation 2.48 with a differential function  $H \in \mathcal{A}$  of an arbitrary order can be reduced to the simplest first-order equation 2.57 via the formal Bäcklund transformation

$$y = h(u + \sum_{i \geq 1} \varepsilon^i \Phi^i), \quad w = x + t h(u + \sum_{i \geq 1} \varepsilon^i \Phi^i),$$

with the coefficients  $\Phi^i$  determined by Equations 2.55 – 2.56. This is a generalization of the well-known fact of the existence of a Lie tangent transformation relating any two scalar partial differential equations of the first order.

#### 2.8.4. Bäcklund via formal Bäcklund transformations

Any finite sum of the series 2.54 gives what is called an *approximate Bäcklund transformation*.

Furthermore, if the series 2.54 breaks off, one gets an *exact Bäcklund transformation*.

If a formal Bäcklund transformation relating Equations 2.48 and 2.53 is known, one can transform infinitesimal symmetries of Equation 2.53 (e.g., the Lie-Bäcklund Symmetries 2.51) into formal and approximate symmetries (or exact Lie-Bäcklund symmetries, provided that the break-off conditions hold) of Equation 2.48. For this, one can use the following *transition formula*:

$$\eta_u = \left[ 1 + \sum_{i \geq 1} \varepsilon^i \Phi_*^i \right]^{-1} \eta_v, \quad (2.58)$$

where  $\eta_u$  and  $\eta_v$  are symmetries of the equations 2.48 and 2.53, respectively, and  $\Phi_*^i$  is the linear differential operator defined in the preamble to Section 2.8.

For applications of this approach, see Chapter 9 of this volume and Baikov, Gazizov, and Ibragimov [1989a].

### 2.8.5. Formal recursions

Symmetries of Equation 2.53 can be constructed recursively by means of the following *recursion operator*,

$$M = \frac{\alpha}{h'(v)} D_x \frac{1}{v_1} + \beta + \gamma v_1 D_x^{-1} h'(v) + \dots, \quad (2.59)$$

previously discussed in this Handbook ([H1], Section 11.3). Here  $\alpha, \beta, \gamma, \dots \in \mathcal{A}$  are arbitrary functions of  $v, x + t h(v), t + 1/(h'(v)v_1), \dots$ . Compare Formula 2.51. For details, see Baikov, Gazizov, and Ibragimov [1989a], Appendix.

The formal Bäcklund transformation 2.54 maps the operator 2.59 into a *formal recursion operator*  $L$  for Equation 2.48. This map is given by the following *transition formula*:

$$L = \left( 1 + \sum_{i \geq 1} \varepsilon^i \Phi_*^i \right)^{-1} M \left( 1 + \sum_{i \geq 1} \varepsilon^i \Phi_*^i \right). \quad (2.60)$$

Any finite sum of the series 2.60 provides an approximate recursion operator. Furthermore, if break-off conditions for the series 2.60 hold,  $L$  is an *exact recursion operator* for Equation 2.48 with any  $\varepsilon$ , e.g., with  $\varepsilon = 1$ .

# 3

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## ***Differential Equations with Distributions: Group Theoretic Treatment of Fundamental Solutions***

This chapter is integrated with Chapter 3 of [H2] and continues the development of Lie group methods for solving initial value problems. Here, the main emphasis is on the construction of fundamental solutions.

The majority of linear differential equations of physical relevance and of obvious mathematical importance have fundamental solutions in the space of distributions (generalized functions). This necessitates the extension of Lie group methods to differential equations in distributions. This natural path of development of Lie group analysis venturing into the space of distributions has been sketched in Ibragimov [1989] and then evolved in Berest [1991] and Ibragimov [1992a,b] (see also Berest [1993], Berest and Ibragimov [1994]).

The group theoretic derivation of fundamental solutions for the Cauchy problem rather than for differential operators has been presented in Ibragimov [1994d]. This recent presentation based on the *Invariance principle for boundary value problems* is adopted in what follows.

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### **3.1. Generalities**

#### **3.1.1. Extension of group transformations to distributions**

Let  $f \in \mathcal{D}'$  be a distribution, i.e., a linear continuous functional over the space  $\mathcal{D}$  of  $C^\infty$  functions with compact supports (test functions). The action of  $f$  on  $\varphi(x) \in \mathcal{D}$ , where  $x = (x^1, \dots, x^n) \in \mathbf{R}^n$ , is denoted by  $\langle f(x), \varphi(x) \rangle$ .

Consider a one-parameter group of transformations in  $\mathbf{R}^n$ :

$$\bar{x} = g(x, a). \quad (3.1)$$

Its infinitesimal transformation is written in the form

$$\bar{x}^i \approx x^i + a\xi^i(x), \quad i = 1, \dots, n. \quad (3.2)$$

The Jacobian of Transformations 3.1,

$$J = \det\left(\frac{\partial \bar{x}^i}{\partial x^j}\right),$$

is positive in a small neighborhood of  $a = 0$ . In what follows, the group parameter  $a$  is assumed to be sufficiently small, so that  $J > 0$ .

**DEFINITION 3.1.** Given a distribution  $f$ , its image  $\bar{f}$  under Transformation 3.1 is defined by the following invariance condition for functionals:

$$\langle f(x), \varphi(x) \rangle = \langle (\bar{f} \circ g^{-1})(\bar{x}), (\varphi \circ g^{-1})(\bar{x}) \rangle. \quad (3.3)$$

Here  $\circ$  denotes the composition of transformations,  $g^{-1}$  is the inversion of  $g$ , and  $\varphi$  is an arbitrary test function.

Consider the regular case, when the distribution  $f$  is defined by a locally integrable function  $f(x)$  as follows:

$$\langle f(x), \varphi(x) \rangle = \int_{\mathbf{R}^n} f(x) \varphi(x) dx.$$

According to the usual change of variables formula in the integral, we have

$$\int_{\mathbf{R}^n} f(x) \varphi(x) dx = \int_{\mathbf{R}^n} (f \circ g^{-1})(\bar{x}) (\varphi \circ g^{-1})(\bar{x}) J^{-1} d\bar{x},$$

that is,

$$\langle f(x), \varphi(x) \rangle = \langle (J^{-1} f \circ g^{-1})(\bar{x}), (\varphi \circ g^{-1})(\bar{x}) \rangle. \quad (3.4)$$

Comparison of Equations 3.3 and 3.4 suggests that  $\bar{f} = J^{-1} f$ .

Thus, we obtain the following extension of the group transformations 3.1 to arbitrary distributions (Ibragimov [1972]):

$$\bar{f} = \left[ \det\left(\frac{\partial \bar{x}^i}{\partial x^j}\right) \right]^{-1} f. \quad (3.5)$$

In particular, for the Dirac distribution  $f = \delta$ , this formula together with the equation

$$\Phi(x)\delta(x) = \Phi(0)\delta(x)$$

where  $\Phi(x)$  is an arbitrary  $C^\infty$  function, yield:

$$\bar{\delta} = \left[ \det \left( \frac{\partial \bar{x}^i}{\partial x^j} \right) \right]_{x=0}^{-1} \delta. \quad (3.6)$$

For the infinitesimal transformation 3.2, the formulas 3.5 and 3.6 reduce, respectively, to the following forms:

$$\bar{f} \approx f - a D_i(\xi^i) f \quad (3.7)$$

and

$$\bar{\delta} \approx \delta - a D_i(\xi^i)|_{x=0} \delta. \quad (3.8)$$

Here

$$D_i(\xi^i) = \sum_{i=1}^n \frac{\partial \xi^i}{\partial x^i}.$$

The usual infinitesimal test for a group invariance of equations given by classical functions applies to equations with distributions as well.

**THEOREM 3.1.** *Let  $F(x, f)$  be a linear function on a distribution  $f$  with smooth coefficients depending on the variables  $x$ . The equation*

$$F(x, f) = 0$$

*is invariant under the group of Transformations 3.1 and 3.5 if and only if it is invariant infinitesimally, i.e., if*

$$F(\bar{x}, \bar{f})|_{F(x, f)=0} = o(a).$$

**REMARK.** For an arbitrary transformation on  $\mathbf{R}^n$  given by

$$\bar{x} = \Phi(x),$$

where  $\Phi$  is a  $C^\infty$  diffeomorphism, the transformation formula 3.5 is written

$$\bar{f} = \left| \det \left( \frac{\partial \bar{x}^i}{\partial x^j} \right) \right|^{-1} f.$$

### 3.1.2. The Leray form and the Dirac measure on hypersurfaces

Consider a hypersurface in  $\mathbf{R}^n$  defined by the equation

$$P(x) = 0,$$

where  $P(x)$  is a continuously differentiable function such that  $\nabla P \neq 0$  on the surface  $P = 0$ .

**DEFINITION 3.2.** *The Leray form for this hypersurface is an  $(n-1)$ -differential form  $\omega$  such that*

$$dP \wedge \omega = dx^1 \wedge \cdots \wedge dx^n.$$

*It can be represented in the form (for any fixed  $i$ )*

$$\omega = (-1)^{i-1} \frac{dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^n}{P_i}$$

*provided that  $P_i \equiv D_i P \neq 0$  (see Leray [1953], Chapter IV, Section 1).*

**DEFINITION 3.3.** *Let  $\theta(P)$  be the Heaviside function, or the characteristic function of the domain  $P \geq 0$ , that is,*

$$\theta(P) = \begin{cases} 1, & P \geq 0, \\ 0, & P < 0. \end{cases}$$

*It defines the distribution*

$$\langle \theta(P), \varphi \rangle = \int_{P \geq 0} \varphi(x) dx.$$

**DEFINITION 3.4.** *The Dirac measure  $\delta(P)$  on the surface  $P(x) = 0$  is defined by*

$$\langle \delta(P), \varphi \rangle = \int_{P=0} \varphi \omega,$$

*where  $\omega$  is the Leray form.*

### 3.1.3. Auxiliary equations

Let us begin with the simplest first-order ordinary differential equation,

$$xf' = 0$$

with one independent variable  $x$ . The only classical solution of this equation is  $f = \text{const.}$ , while its general solution in distributions depends on two arbitrary constants and has the form

$$f = C_1 \theta(x) + C_2.$$

We will use the following natural generalization of this equation. It is known that the distributions  $\theta(P)$  and  $\delta(P)$  given by Definitions 3.3 and 3.4 satisfy the equations (see, e.g., Gel'fand and Shilov [1959]):

$$\theta'(P) = \delta(P),$$

$$P\delta(P) = 0,$$

$$P\delta^{(m)}(P) + m\delta^{(m-1)}(P) = 0, \quad m = 1, 2, \dots,$$

where  $\delta^{(m)}$  is the  $m$ th derivative of  $\delta(P)$  with respect to  $P$ . It follows that the first-order differential equation

$$Pf'(P) + mf(P) = 0 \tag{3.9}$$

has the general solution in distributions given by

$$f = C_1 \theta(P) + C_2, \quad \text{for } m = 0, \tag{3.10}$$

$$f = C_1 \delta^{(m-1)}(P) + C_2 P^{-m}, \quad \text{for } m = 1, 2, \dots, \tag{3.11}$$

where  $C_1$  and  $C_2$  are arbitrary constants.

### 3.1.4. Invariance principle

Given a linear partial differential operator  $L$ , consider a boundary value (in particular, initial value) problem,

$$Lu = F(x), \tag{3.12}$$

$$u|_S = h(x) \tag{3.13}$$

with the data  $h(x)$  defined on the manifold  $S \subset \mathbf{R}^n$ .

**DEFINITION 3.5.** The problem 3.12–3.13 is said to be invariant under a group  $G$  if the following hold:

1) the differential equation 3.12 admits  $G$ ,

2) the manifold  $S$  together with Equation 3.13 are invariant under the group  $G$ .

**The invariance principle.** Let the boundary value problem 3.12 – 3.13 be invariant under the group  $G$ . Then we should seek the solution among the functions invariant under  $G$ .

**Example 1.** Consider the following boundary value problem in the circle  $r \leq 1$ :

$$u_{xx} + u_{yy} = e^u, \quad (3.14)$$

$$u|_{r=1} = 0, \quad (3.15)$$

where  $r = \sqrt{x^2 + y^2}$ .

The Liouville equation 3.14 admits the group of conformal transformations on the  $(x, y)$  plane properly extended to the variable  $u$ . The infinitesimal generator of this group is of the form

$$X = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} - 2\xi_x \frac{\partial}{\partial u},$$

where the coefficients  $\xi(x, y)$  and  $\eta(x, y)$  are arbitrary analytic functions, i.e., defined by the Cauchy-Riemann equations:

$$\xi_x - \eta_y = 0, \quad \xi_y + \eta_x = 0.$$

It is convenient to investigate the invariance of the boundary condition 3.15 in the polar coordinates  $(r, \psi)$ :

$$x = r \cos \psi, \quad y = r \sin \psi.$$

Then the symmetry operator for the Liouville equation

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\psi\psi} = e^u$$

has the form

$$X = \alpha(r, \psi) \frac{\partial}{\partial r} + \beta(r, \psi) \frac{\partial}{\partial \psi} - 2\alpha_r \frac{\partial}{\partial u}.$$

Its coefficients are determined by the system of Cauchy-Riemann equations in polar coordinates:

$$\beta_r + \frac{1}{r^2}\alpha_\psi = 0, \quad \beta_\psi - \alpha_r + \frac{\alpha}{r} = 0.$$

One obtains from this system, by eliminating  $\beta$ , the second-order equation for  $\alpha(r, \psi)$ :

$$\alpha_{rr} + \frac{1}{r^2} \alpha_{\psi\psi} - \frac{1}{r} \alpha_r + \frac{1}{r^2} \alpha = 0.$$

The infinitesimal invariance condition of the boundary manifold  $r = 1$  and of Equation 3.15 yield:

$$\alpha|_{r=1} = 0, \quad \alpha_r|_{r=1} = 0.$$

It follows from the Cauchy-Kovalevskaya theorem that  $\alpha = 0$  in a neighborhood of  $r = 1$ . Then the Cauchy-Riemann system yields  $\beta = \text{const.}$

Thus, the boundary value problem 3.14 – 3.15 admits the operator

$$X_1 = \frac{\partial}{\partial \psi},$$

i.e., the problem is invariant under the one-parameter group of rotations. According to the invariance principle, the solution is taken in the form  $u = U(r)$ . Then Equation 3.14 reduces to the ordinary differential equation

$$U'' + \frac{1}{r} U' - e^U = 0. \quad (3.16)$$

We will consider the solutions bounded at the “singular” point  $r = 0$ . Taking into account Equation 3.15, we have the following side conditions:

$$U(1) = 0, \quad |U(0)| < \infty.$$

Equation 3.16 admits the two-dimensional Lie algebra spanned by

$$Y_1 = r \frac{\partial}{\partial r} - 2 \frac{\partial}{\partial U}, \quad Y_2 = r \ln r \frac{\partial}{\partial r} - 2(1 + \ln r) \frac{\partial}{\partial U}.$$

In accordance with Lie’s integration algorithm (see [H1], Section 2.2.2), we find the transformation of the symmetry algebra to its canonical form. This transformation is given by the change of variables:

$$t = \ln r, \quad v = U + 2 \ln r.$$

Equation 3.16 is rewritten, for the function  $v(t)$ , in the integrable form:

$$v'' = e^v.$$

dimensional subalgebra spanned by

$$X_0 = \frac{\partial}{\partial t}, \quad X_i = \frac{\partial}{\partial x^i}, \quad X_{ij} = x^j \frac{\partial}{\partial x^i} - x^i \frac{\partial}{\partial x^j}, \quad X_{0i} = 2t \frac{\partial}{\partial x^i} - x^i u \frac{\partial}{\partial u},$$

$$Z_1 = 2t \frac{\partial}{\partial t} + x^i \frac{\partial}{\partial x^i}, \quad Z_2 = u \frac{\partial}{\partial u}, \quad Y = t^2 \frac{\partial}{\partial t} + tx^i \frac{\partial}{\partial x^i} - \frac{1}{4}(2nt + |x|^2)u \frac{\partial}{\partial u}$$

and the infinite-dimensional ideal consisting of the operators

$$X_\tau = \tau(t, x) \frac{\partial}{\partial u},$$

where  $\tau(t, x)$  is an arbitrary solution of Equation 3.17.

In what follows, we let  $\tau(t, x) = 0$  and use only the  $[\frac{1}{2}(n+1)(n+2)+3]$ -dimensional symmetry algebra spanned by

$$X_0, X_i, X_{ij}, X_{0i}, Z_1, Z_2, Y, \quad i, j = 1, \dots, n. \quad (3.18)$$

### 3.2.2. The symmetry algebra of the initial value problem for the fundamental solution

The theory of distributions reduces an arbitrary Cauchy problem for the heat equation to determining the following *fundamental solution*.

**DEFINITION 3.6.** The distribution  $u = E(t, x)$  is called the *fundamental solution of the Cauchy problem for the heat equation* if it solves Equation 3.17 for  $t > 0$  and satisfies the initial condition

$$u|_{t=0} = \delta(x), \quad (3.19)$$

where  $u|_{t=0} \equiv \lim_{t \rightarrow +0} E(t, x)$ , and  $\delta(x)$  is the Dirac distribution at  $x = 0$ .

**THEOREM 3.2.** Let  $L$  be the Lie algebra with the basis 3.18, and  $K$  its maximal subalgebra admitted by the initial value problem 3.17, 3.19. Then  $K$  is the linear span of the operators

$$X_{ij}, X_{0i}, Z_1 - nZ_2, Y, \quad i, j = 1, \dots, n. \quad (3.20)$$

**PROOF.** Since the algebra  $L$  is admitted by the differential equation 3.17, we shall consider only the invariance condition 2 of Definition 3.5.

In our case, the initial manifold  $S$  is given by  $t = 0$ . Further, the invariance of the initial data 3.19 requires, in particular, that the support of  $\delta(x)$ , i.e., the point  $x = 0$ , be unaltered. Thus, Definition 3.5 requires that the system of equations  $t = 0, x = 0$  be invariant. This reduces the algebra  $L$  by the translation operators  $X_i, X_0$ .

Hence, the operators 3.18 are restricted to the following:

$$X_{ij}, X_{0i}, Z_1, Z_2, Y.$$

Equation 3.19 is invariant under the operators  $X_{ij}, X_{0i}$ , and  $Y$ . It is not invariant under the two-dimensional algebra spanned by  $Z_1, Z_2$ . Therefore, we inspect the infinitesimal invariance test for the linear combination

$$(Z_1 + kZ_2)|_{t=0} = x^i \frac{\partial}{\partial x^i} + ku \frac{\partial}{\partial u}, \quad k = \text{const.}$$

Under this operator, the variable  $u$  and the  $\delta$ -function are subjected to the infinitesimal transformations (see Equation 3.8):

$$\bar{u} \approx u + aku, \quad \bar{\delta} \approx \delta - an\delta.$$

It follows that  $\bar{u} - \bar{\delta} = u - \delta + a(ku + n\delta) + o(a)$  and that

$$(\bar{u} - \bar{\delta})|_{u=\delta} = a(k + n)\delta + o(a).$$

According to Theorem 3.1, the invariance condition of Equation 3.19 has the form  $k + n = 0$ .

Thus, we arrive at the algebra  $K$  spanned by the operators 3.20.

### 3.2.3. Derivation of the fundamental solution from the invariance principle

**THEOREM 3.3.** *The fundamental solution of the Cauchy problem for the heat equation,*

$$E(t, x) = (2\sqrt{\pi t})^{-n} \exp[-|x|^2/(4t)], \quad (3.21)$$

*is uniquely determined by the invariance principle. Namely, it is the only function  $u = \phi(t, x)$  which satisfies the initial condition 3.19 and is invariant under the group of rotations, Galilean transformations, and dilations with the infinitesimal generators*

$$X_{ij}, \quad X_{0i}, \quad Z_1 - nZ_2, \quad i, j = 1, \dots, n. \quad (3.22)$$

**PROOF.** We first notice that the functionally independent invariants of the rotations are  $t, r, u$ , where

$$r = |x| \equiv \sqrt{(x^1)^2 + \cdots + (x^n)^2}.$$

Then we write the restriction of the Galilean operators  $X_{0i}$  to functions of these invariants as follows:

$$X_{0i} = x^i \left( 2 \frac{t}{r} \frac{\partial}{\partial r} - u \frac{\partial}{\partial u} \right).$$

For these operators, the independent invariants are  $t$  and  $p = u \exp[r^2/(4t)]$ . The last operator 3.22 is written in these variables in the form:

$$Z_1 - nZ_2 = 2t \frac{\partial}{\partial t} - np \frac{\partial}{\partial p}.$$

It has the only independent invariant  $J = t^{n/2} p$ . Hence, the function

$$J = (\sqrt{t})^n u \exp[r^2/(4t)]$$

is the only common invariant for the operators 3.22.

It follows that the general form of the function  $u = \phi(t, x)$  invariant under the operators 3.22 is given by  $J = C$ , or by

$$u = C(\sqrt{t})^{-n} \exp[-r^2/(4t)], \quad C = \text{const.}$$

In view of the known formula

$$\lim_{t \rightarrow +0} \left( (\sqrt{t})^{-n} \exp[-|x|^2/(4t)] \right) = (2\sqrt{\pi})^n \delta(x),$$

the initial condition 3.19 yields  $C = (2\sqrt{\pi})^{-n}$ .

Thus, we have obtained (uniquely) the fundamental solution 3.21.

**REMARK 1.** One can readily verify that the function 3.21 is also invariant under the operator  $Y$  from Basis 3.20, i.e., admits the Lie algebra  $K$ . However, we do not need here this *excess symmetry* of the fundamental solution.

**REMARK 2.** The fundamental solution  $\mathcal{E}$  of the heat operator, i.e., the solution of the equation

$$\left( \frac{\partial}{\partial t} - \Delta \right) \mathcal{E} = \delta(t, x),$$

is obtained by the formula  $\mathcal{E} = \theta(t)E$ , where  $\theta(t)$  is the Heaviside function. Hence,

$$\mathcal{E} = \theta(t)(2\sqrt{\pi t})^{-n} \exp[-|x|^2/(4t)].$$

### 3.2.4. Solution of the Cauchy problem

The solution  $u(t, x)$  of the Cauchy problem for Equation 3.17 with an arbitrary initial data,

$$u|_{t=0} = u_0(x)$$

is given by the convolution of the data and the fundamental solution 3.21:

$$u(t, x) = E * u_0 \equiv \int_{\mathbf{R}^n} u_0(y) E(t, x - y) dy.$$

Hence,

$$u(t, x) = (2\sqrt{\pi t})^{-n} \int_{\mathbf{R}^n} u_0(y) \exp(-|x - y|^2/(4t)) dy, \quad t > 0.$$

## 3.3. Wave equation

### 3.3.1. Symmetry algebra of the equation

Consider the wave equation with  $n > 1$  spatial variables:

$$u_{tt} = \Delta u \tag{3.23}$$

where  $\Delta$  is the  $n$ -dimensional Laplacian in the variables  $x = (x^1, \dots, x^n) \in \mathbf{R}^n$ .

The maximal Lie algebra admitted by Equation 3.23 is composed of the finite-dimensional subalgebra spanned by

$$\begin{aligned} X_0 &= \frac{\partial}{\partial t}, \quad X_i = \frac{\partial}{\partial x^i}, \quad X_{ij} = x^j \frac{\partial}{\partial x^i} - x^i \frac{\partial}{\partial x^j}, \quad X_{0i} = t \frac{\partial}{\partial x^i} + x^i \frac{\partial}{\partial t}, \\ Z_1 &= t \frac{\partial}{\partial t} + x^i \frac{\partial}{\partial x^i}, \quad Z_2 = u \frac{\partial}{\partial u}, \quad Y_0 = (t^2 + |x|^2) \frac{\partial}{\partial t} + 2tx^i \frac{\partial}{\partial x^i} - (n-1)tu \frac{\partial}{\partial u}, \\ Y_i &= 2tx^i \frac{\partial}{\partial t} + (2x^i x^j + (t^2 - |x|^2)\delta^{ij}) \frac{\partial}{\partial x^j} - (n-1)x^i u \frac{\partial}{\partial u}, \quad i, j = 1, \dots, n, \end{aligned}$$

and the infinite-dimensional ideal consisting of the operators

$$X_\tau = \tau(t, x) \frac{\partial}{\partial u},$$

where  $\tau(t, x)$  is an arbitrary solution of Equation 3.23.

In what follows, we let  $\tau(t, x) = 0$  and use only the  $[\frac{1}{2}(n+2)(n+3)+1]$ -dimensional symmetry algebra spanned by

$$X_0, X_i, X_{ij}, X_{0i}, Z_1, Z_2, Y_0, Y_i, \quad i, j = 1, \dots, n. \quad (3.24)$$

### 3.3.2. Fundamental solution of the Cauchy problem

**LEMMA 3.1.** *The Cauchy problem with arbitrary initial conditions,*

$$u_{tt} - \Delta u = 0, \quad t > 0,$$

$$u|_{t=0} = u_0(x), \quad u_t|_{t=0} = u_1(x) \quad (3.25)$$

*reduces to the following special Cauchy problem:*

$$u_{tt} - \Delta u = 0, \quad u|_{t=0} = 0, \quad u_t|_{t=0} = h(x). \quad (3.26)$$

**PROOF.** Let  $v(t, x)$  and  $w(t, x)$  be the solutions of the problem 3.26 with  $h(x) = u_0(x)$  and  $h(x) = u_1(x)$ , respectively. Then the solution  $u(t, x)$  to the general problem 3.25 is given by

$$u(t, x) = w(t, x) + \frac{\partial v(t, x)}{\partial t}. \quad (3.27)$$

**DEFINITION 3.7.** *The distribution  $u = E(t, x)$  is called the fundamental solution of the Cauchy problem for the wave equation if it solves Equation 3.23 for  $t > 0$  and satisfies the initial conditions*

$$u|_{t=0} = 0 \quad (3.28)$$

and

$$u_t|_{t=0} = \delta(x), \quad (3.29)$$

where

$$u|_{t=0} \equiv \lim_{t \rightarrow +0} E(t, x), \quad u_t|_{t=0} \equiv \lim_{t \rightarrow +0} \frac{\partial E(t, x)}{\partial t}.$$

**REMARK.** The solution  $u(t, x)$  to the problem 3.26 is given by the convolution (in the spatial variables  $x$ ) of the data with the fundamental solution:

$$u(t, x) = E * h(x).$$

### 3.3.3. The symmetry of the initial data

**THEOREM 3.4.** *Let  $L$  be the Lie algebra with the basis 3.24, and  $K$  its maximal subalgebra admitted by the initial conditions 3.28 and 3.29. Then  $K$  is the linear span of the operators*

$$X_{ij}, X_{0i}, Z_1 + (1 - n)Z_2, Y_0, Y_i, \quad i, j = 1, \dots, n. \quad (3.30)$$

**PROOF.** In accordance with Definition 3.5, we require the invariance of the equations  $t = 0, x = 0$  (see the proof of Theorem 3.2). Under this requirement, the operators 3.24 are restricted to the generators of rotations, Lorentz transformations, dilations, and conformal transformations:

$$X_{ij}, X_{0i}, Z_1, Z_2, Y_0, Y_i. \quad (3.31)$$

It is easy to see that Equation 3.28 is invariant under the operators 3.31. Therefore we shall inspect the invariance of Equation 3.29 only. It is obvious that Equation 3.29 is invariant with respect to the rotations.

Turning to the Lorentz transformations, consider their generators  $X_{0i}$  in the prolonged form:

$$X_{0i} = t \frac{\partial}{\partial x^i} + x^i \frac{\partial}{\partial t} - u_i \frac{\partial}{\partial u_t}.$$

Under these operators, the variable  $u_t$  and the  $\delta$ -function are subjected to the following infinitesimal transformations:

$$\bar{u}_t \approx u_t - a u_i, \quad \bar{\delta} \approx \delta.$$

In view of Equation 3.28, we have  $u_i|_{t=0} = 0$ . Hence,  $\bar{u}_t \approx u_t$  at  $t = 0$ . Hence, Equation 3.29 is Lorentz invariant.

Likewise, one can verify that Equation 3.29 is invariant under the conformal transformations with the generators  $Y_0, Y_i$ .

Finally, consider the linear combination of the remaining operators 3.31 prolonged to  $u_t$ :

$$Z_1 + kZ_2 = t \frac{\partial}{\partial t} + x^i \frac{\partial}{\partial x^i} + k u \frac{\partial}{\partial u} + (k - 1) u_t \frac{\partial}{\partial u_t}.$$

It follows that

$$\bar{u}_t \approx u_t + a(k-1)u_t, \quad \bar{\delta} \approx \delta - an\delta.$$

Hence, Theorem 3.1 yields  $k = 1 - n$ .

### 3.3.4. Derivation of the fundamental solution from the invariance principle

We discuss here the case when  $n$  is odd. The fundamental solution for the wave equations with even  $n$  is easily obtained by Hadamard's method of descent (Hadamard [1923]) and has been presented in [H2], Chapter 7.

**THEOREM 3.5.** *The fundamental solution of the Cauchy problem for the wave equation 3.23 with an odd number  $n$  of spatial variables has the form:*

$$E(x, t) = \begin{cases} \frac{1}{2}\pi^{\frac{1-n}{2}}\delta^{(\frac{n-3}{2})}(\Gamma), & n \geq 3, \\ \frac{1}{2}\theta(\Gamma), & n = 1, \end{cases} \quad (3.32)$$

where

$$\Gamma = t^2 - |x|^2. \quad (3.33)$$

It is determined uniquely by the invariance principle. Namely,  $E(t, x)$  given by Equation 3.32 is the only distribution which satisfies the initial conditions 3.28 and 3.29 and is invariant under the group of rotations, Lorentz transformations, and dilations with the infinitesimal generators

$$X_{ij}, \quad X_{0i}, \quad Z_1 + (1-n)Z_2, \quad i, j = 1, \dots, n. \quad (3.34)$$

**PROOF.** (Cf. Proof of Theorem 3.3.) We first find a basis of invariants for the generators  $X_{ij}, X_{0i}$  of the isometric motions (rotations and Lorentz transformations). In the space of the variables  $(t, x, u)$ , we have the following two independent invariants:

$$u, \quad \Gamma = t^2 - |x|^2.$$

Then we write the restriction of the last operator 3.34 to functions of these invariants as follows:

$$Z_1 + (1-n)Z_2 = 2\Gamma \frac{\partial}{\partial \Gamma} + (1-n)u \frac{\partial}{\partial u}. \quad (3.35)$$

Now, let us look for invariant distributions of the form

$$u = f(\Gamma).$$

### 3.4. Wave equations with nontrivial conformal group

The classical wave equation 3.23 is an example of the equations mentioned in the title. We will show here that *the invariance principle* applies to all the wave equations with nontrivial conformal group.

The definition of the wave equation in a curved space-time as a conformally invariant equation is due to Penrose [1964] (a discussion of general properties) and to Ibragimov [1968], [1969a] (uniqueness of the conformally invariant equation in any space-time with nontrivial conformal group and non-uniqueness in all other space-times).

The solution of the Cauchy problem for the equations under consideration has been given in Ibragimov and Mamontov [1970], [1977] by adapting the Fourier transformation approach. The Lie group approach to the Cauchy problem and to the Huygens principle has been suggested in Ibragimov [1970]. The fundamental solution has been first constructed in Mamontov [1984] by an *ad hoc* method, and then discussed from the group theoretic viewpoint in Berest [1991] and Berest and Ibragimov [1994].

The reader interested in details of the conformal group analysis of the wave equation in Riemannian spaces is referred to Ibragimov [1983]. See also Chapters 4 and 7 of [H2], and Ibragimov [1992c]. For the notation and terminology used in this Section, see [H2], Section 3.1.

#### 3.4.1. Riemannian spaces with nontrivial conformal group

Consider  $(n + 1)$ -dimensional Riemannian spaces  $V_{(n+1)}$  of the hyperbolic signature  $(+ - \cdots -)$  given by the fundamental metric forms

$$ds^2 = g_{ij}(x)dx^i dx^j.$$

Here  $i, j = 0, 1, \dots, n$ . We will assume in what follows that  $n \geq 3$ .

**DEFINITION 3.8.** A space  $V_{(n+1)}$  is said to be a Riemannian space with nontrivial conformal group if, in  $V_{(n+1)}$  and in every space conformal to  $V_{(n+1)}$ , the maximal group of conformal transformations does not reduce to the group of isometric motions.

**THEOREM 3.6.**  $V_{(n+1)}$  is a space with nontrivial conformal group if and only

if it is conformal to a space with the plane-wave metric:

$$ds^2 = (dx^0)^2 - (dx^1)^2 - \sum_{i,j=2}^n a_{ij}(x^1 - x^0) dx^i dx^j. \quad (3.36)$$

Here,  $[a_{ij}]$  is an arbitrary positive definite matrix with entries depending on a single variable  $x^1 - x^0$ .

This result is due to Bilyalov [1963] (for  $n = 3$ ) and to Chupakhin [1979] (for  $n > 3$ ). Detailed discussions are to be found in Petrov [1966], Chapter VII, and in Ibragimov [1983], Section 8.5.

### 3.4.2. The wave equation in $V_{(n+1)}$ and its symmetry algebra

**THEOREM 3.7.** *Let  $V_{(n+1)}$  be a space with nontrivial conformal group. Then any second-order linear differential equation in  $V_{(n+1)}$ ,*

$$g^{ij}(x)u_{ij} + b^i(x)u_i + c(x)u = 0,$$

*admitting the maximal group of conformal motions in  $V_{(n+1)}$  is equivalent to the following equation:*

$$\Delta_2 u + \frac{n-1}{4n} R u = 0. \quad (3.37)$$

Here,  $g_{ik}g^{kj} = \delta_i^j$ ,  $R$  is the scalar curvature of  $V_{(n+1)}$ , and  $\Delta_2 u$  is Beltrami's second-order differential parameter defined by

$$\Delta_2 u = g^{ij}(u_{ij} - \Gamma_{ij}^k u_k)$$

where  $u_k$  and  $u_{ij}$  denote the partial derivatives of the first and the second order, respectively, and  $\Gamma_{ij}^k$  are the Christoffel symbols.

Thus, in the spaces with nontrivial conformal group, Equation 3.37 is the only *conformally invariant* equation (up to equivalence transformations defined in [H2], Section 3.1.2); other equations admit only a subgroup of the conformal group. Another important property of this equation, related to the Huygens principle, underscores its physical significance.

In spaces with trivial conformal group, Equation 3.37 is also conformally invariant. However, it is not unique.

reduces to the Cauchy problem with the following special initial data:

$$u|_{t=0} = 0, \quad u_t|_{t=0} = h(x). \quad (3.42)$$

Here,

$$x = (x^1, \dots, x^n).$$

**PROOF.** Let  $v(t, x)$  and  $w(t, x)$  be the solutions of the special Cauchy problem with

$$h(x) = u_0(x)$$

and

$$h(x) = \frac{\partial u_0(x)}{\partial x} - u_1(x),$$

respectively. Then the function

$$u(t, x) = \frac{\partial v(t, x)}{\partial t} + \frac{\partial v(t, x)}{\partial x} - w(t, x) \quad (3.43)$$

solves Equation 3.38 and satisfies the initial conditions 3.41.

**DEFINITION 3.10.** The distribution  $u = E(t, x; t_0, x_0)$  is called the fundamental solution of the Cauchy problem for the wave equation 3.38, at the point  $(t_0, x_0)$ , if it solves Equation 3.38 for  $t > t_0$  and satisfies the initial conditions

$$u|_{t=t_0} = 0, \quad u_t|_{t=t_0} = \delta(x - x_0). \quad (3.44)$$

### 3.4.4. The symmetry of the initial data

**THEOREM 3.9.** Let  $L$  be the Lie algebra with the basis 3.39, and  $K$  its maximal subalgebra admitted by the initial conditions 3.44. Then  $K$  is the linear span of the following  $n$  operators:

$$\tilde{Y}_i = (x^i - x_0^i) \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial x^1} \right) - \sum_{j=2}^n \left( A^{ij}(x^1 - t) - A^{ij}(x_0^1 - t_0) \right) \frac{\partial}{\partial x^j}, \quad i = 2, \dots, n,$$

$$Z = (t - t_0 + x^1 - x_0^1) \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial x^1} \right) + \sum_{k=2}^n (x^k - x_0^k) \frac{\partial}{\partial x^k} + (1 - n)u \frac{\partial}{\partial u}. \quad (3.45)$$

**PROOF.** The natural adaptation of the proof of Theorem 3.4 applies here.

### 3.4.5. Derivation of the fundamental solution from the invariance principle

**THEOREM 3.10.** *The fundamental solution of the Cauchy problem for the wave equation 3.38 with an odd  $n \geq 3$  has the form:*

$$E(t, x; t_0, x_0) = \frac{1}{2} \pi^{\frac{1-n}{2}} h(\sigma) \delta^{(\frac{n-3}{2})}(\Gamma). \quad (3.46)$$

Here,

$$\Gamma = (t - t_0)^2 - (x^1 - x_0^1)^2 - (t - t_0 - x^1 + x_0^1) \sum_{i,j=2}^n \tilde{A}_{ij} (x^i - x_0^i)(x^j - x_0^j) \quad (3.47)$$

is the geodesic distance between the points  $(t, x) \in V_{(n+1)}$  and  $(t_0, x_0) \in V_{(n+1)}$ , and

$$h(\sigma) = \left[ \frac{|\sigma - \sigma_0|^{n-1}}{|\det \|\tilde{A}^{ij}(\sigma)\||} \right]^{\frac{1}{2}}, \quad (3.48)$$

where

$$\|\tilde{A}_{ij}\| = \|A^{ij}(x^1 - t) - A^{ij}(x_0^1 - t_0)\|^{-1}.$$

It is determined uniquely by the invariance principle. Namely,  $E(t, x)$  given by Equation 3.46 is the only distribution which satisfies the initial conditions 3.42 and is invariant under the  $n$ -parameter group with the infinitesimal generators 3.45.

**PROOF.** (See Proof of Theorem 3.5.) A basis of invariants for the generators  $\tilde{Y}_i$  of isometric motions in  $V_{(n+1)}$  consists of the following three independent invariants:

$$u, \quad \Gamma, \quad \sigma = x^1 - t,$$

where  $\Gamma$  is given by Equation 3.47 (its calculation is found in Ibragimov [1983], Section 12.1). Therefore, we look for the distributions, invariant under the operator  $Z$  of Basis 3.45, given in the general form:

$$u = f(\Gamma, \sigma).$$

We note that  $\sigma$  is an invariant for the operator  $Z$  and that  $Z(\Gamma) = 2\Gamma$ . It follows that the invariance condition under the operator  $Z$  is written as the first-order differential equation obtained in Section 3.3.4:

$$2\Gamma \frac{\partial f}{\partial \Gamma} + (n-1)f(\Gamma) = 0. \quad (3.49)$$

In accordance with Section 3.3.4, the general solution of Equation 3.49 is given by

$$u = C_1(\sigma)\delta^{(\frac{n-3}{2})}(\Gamma) + C_2(\sigma)\Gamma^{\frac{1-n}{2}}$$

with arbitrary functions  $C_1(\sigma)$  and  $C_2(\sigma)$ . The first initial condition 3.42 yields  $C_2(\sigma) = 0$  (see Section 3.3.4), while the second condition 3.42 leads to

$$C_1(\sigma) = \frac{1}{2}\pi^{\frac{1-n}{2}}h(\sigma),$$

where  $h(\sigma)$  is given by Equation 3.48. Thus, we have obtained (uniquely) the fundamental solution 3.46.

**THEOREM 3.11.** *The fundamental solution of the Cauchy problem for the wave equation 3.38 with an even  $n > 2$  is uniquely determined by the invariance principle and has the form:*

$$E(t, x; t_0, x_0) = \frac{1}{2}\pi^{-\frac{n}{2}}h(\sigma)\left(\frac{\theta(\Gamma)}{\sqrt{\Gamma}}\right)^{(\frac{n-2}{2})}. \quad (3.50)$$

**PROOF.** The proof is similar to the case of odd  $n$ . The formula 3.49 is also obtained by the method of descent (for details of an application of this method to the solution of the Cauchy problem, see Ibragimov and Mamontov [1977] or Ibragimov [1983], Section 13.3).

## Recursions

The object of this chapter<sup>1</sup> is a discussion of newly developed methods for obtaining infinitely many Lie-Bäcklund (see Chapter 1) and generalized symmetries of partial differential equations.

An algorithmic way to construct an infinite (countable) set of symmetries is to derive a recursion relation between symmetries. Usually, recursions are linear with respect to symmetries and hence, they are given by linear operators. These linear operators are known in the literature as *squared eigenfunction operators* (Ablowitz, Kaup, Newell, and Segur [1974]), *recursion operators* (Olver [1977], [1986], Ibragimov and Shabat [1979], [1980a], Ibragimov [1983]), *strong or hereditary symmetries* (Fuchssteiner [1979], Zakharov and Konopelchenko [1984]), *Kähler operators* (Magri [1978]), and *regular operators* (Gel'fand and Dorfman [1979], [1980]).

A method of finding linear and multilinear recursions is provided by the master symmetries approach (Fuchssteiner [1983]). This approach utilizes the structure of a Lie algebra of vector fields associated with partial differential equations.

A recursion operator plays at least two additional important roles. First, it generates infinite hierarchies of equations integrable by a spectral problem in the context of the inverse scattering transform method. Second, it generates, in the case of Hamiltonian equations, infinite sets of Hamiltonian structures associated with these equations.

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### 4.1. Notation

(Cf. Chapter 1)

Consider evolution (system of) equations of the form

$$u_t = K(u, x, t). \quad (4.1)$$

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Here,  $x = (x^1, \dots, x^d)$ ,  $u = (u^1, \dots, u^n)$  is an element of the linear space  $S$  of  $n$ -dimensional (real) vector-functions  $u(x, t)$  vanishing rapidly at  $|x| \rightarrow \infty$ , and  $K : S \rightarrow S$  is a  $C^\infty$ -map that may depend explicitly on  $x$  and  $t$ .

The derivative of  $K$  in the direction  $v \in S$  is the Fréchet derivative:

$$K'(u, x, t)[v] = \frac{\partial}{\partial \epsilon} K(u + \epsilon v, x, t) \Big|_{\epsilon=0}. \quad (4.2)$$

Let  $K = (K_1, \dots, K_n)$  be a vector such that its components  $K_\alpha$  are differential functions (see Chapter 1):

$$K_\alpha = K_\alpha(u, u_{(1)}, \dots, u_{(m)}), \quad \alpha = 1, \dots, n,$$

where  $u_{(k)}$  denotes the set of  $k$ -order partial derivatives of  $u^\alpha$  with respect to  $x^i$ . Then the Formula 4.2 coincides with the expressions  $K_*(v, x, t)$  of Ibragimov [1983] and  $D_k(v, x, t)$  of Olver [1986], viz.

$$(K')_{\alpha\beta} \equiv (D_k)_{\alpha\beta} \equiv (K_*)_{\alpha\beta} = \sum_J \left( \frac{\partial K_\alpha}{\partial u_J^\beta} \right) D_J, \quad \alpha, \beta = 1, \dots, m. \quad (4.3)$$

Here,  $J = (j_1, \dots, j_l)$  is a multi-index with  $1 \leq j_s \leq n$ ,

$$u_J^\beta(x, t) = \frac{\partial^l u^\beta(x, t)}{\partial x^{j_1} \dots \partial x^{j_l}},$$

and

$$D_J = D_{j_1} \dots D_{j_l},$$

where  $D_{j_s}$  is the total derivative with respect to  $x^{j_s}$  (cf. Chapter 1, Formula 1.2). The summation in Equation 4.3 extends up to  $l = k$ .

**Example 1.** Korteweg-de Vries equation ( $d = 1$ ):

$$u_t = u_{xxx} + 6uu_x. \quad (4.4)$$

Here,  $K(u) = u_{xxx} + 6uu_x$ , and according to Formula 4.3,

$$K'(u) = D_x^3 + 6u D_x + 6u_x.$$

The following example illustrates the abstract form of a system 4.1 where  $K(u)$  is not a differential function.

**Example 2.** Benjamin-Ono equation ( $d = 1$ ):

$$u_t = Hu_{xx} + 2uu_x. \quad (4.5)$$

Here,  $H$  is the Hilbert transform given by

$$(Hf)(x) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{f(\xi)}{\xi - x} d\xi, \quad (4.6)$$

where the principal value of the integral is implied. In this example,  $K(u) = Hu_{xx} + 2uu_x$ , and according to Formula 4.2,

$$K'[v] = Hv_{xx} + 2(uv)_x.$$

For functions  $\gamma(u, x, t)$  and  $\sigma(u, x, t)$  defined on  $S$  and vanishing sufficiently fast for  $|x| \rightarrow \infty$ , the scalar product is given by

$$\langle \gamma, \sigma \rangle = \int_{\mathbf{R}^d} \gamma(u(x, t), x, t) \sigma(u(x, t), x, t) dx. \quad (4.7)$$

Let  $S^*$  be the dual space to  $S$  with respect to the bilinear form 4.7. Thus, for  $\gamma \in S^*$  and  $\sigma \in S$ , the expression  $\langle \gamma, \sigma \rangle$  is an application of the linear functional  $\gamma$  to  $\sigma$ .

Given an operator  $T: S \rightarrow S$ , the adjoint (transpose) operator  $T^+ : S^* \rightarrow S^*$  is defined by the equation

$$\langle a, Tb \rangle = \langle T^+ a, b \rangle$$

for all  $a \in S^*, b \in S$ . An operator  $T$  is called symmetric if

$$\langle a, Tb \rangle = \langle Ta, b \rangle,$$

and skew-symmetric if

$$\langle a, Tb \rangle = -\langle Ta, b \rangle$$

for all  $a \in S^*, b \in S$ .

An operator  $\theta: S^* \rightarrow S$  is called symmetric if

$$\langle a, \theta b \rangle = \langle b, \theta a \rangle,$$

and skew-symmetric if

$$\langle a, \theta b \rangle = -\langle b, \theta a \rangle$$

for all  $a, b \in S^*$ . For operators  $J: S \rightarrow S^*$  the definitions are similar.

A function  $\gamma: S \rightarrow S^*$  (it may depend explicitly on  $x, t$ ) is said to be a *gradient* if it has a *potential*  $P$ , i.e., a map  $P: S \rightarrow \mathbb{R}$  such that

$$\langle \gamma(u, x, t), v \rangle = P'(u, x, t)[v] \quad (4.8)$$

for all  $u, v \in S$ . The functional  $P$  may depend explicitly on  $x, t$ . We write  $\gamma = \text{grad} P$ . Thus, the equation

$$\langle \text{grad} P, v \rangle = P'[v] \quad (4.8')$$

is a definition of a gradient of  $P(u, x, t)$  with respect to  $u$ .

A function  $\gamma(u, x, t)$  is a gradient, iff

$$\gamma'^+ = \gamma',$$

i.e., if the linear operator  $\gamma'(u, x, t)[\cdot]$  is symmetric. Then the potential  $P$  of  $\gamma$  is given by the homotopy formula

$$P(u, x, t) = \int_0^1 \langle \gamma(\lambda u, x, t), u \rangle d\lambda. \quad (4.9)$$

In the case of classical variational calculus, the functional  $P$  has the form

$$P = \int_{\mathbb{R}^d} \rho(u, u_{(1)}, \dots, u_{(m)}, x, t) dx, \quad (4.10)$$

where  $\rho$  (a *Lagrangian*) is a differential function. In this particular case, the gradient coincides with the variational derivative of  $P$ :

$$\gamma = \text{grad} P = \frac{\delta P}{\delta u^\alpha}, \quad (4.11)$$

where (in the notation used in Formula 4.3)

$$\frac{\delta}{\delta u^\alpha} = \sum_J (-D)_J \frac{\partial}{\partial u_J^\alpha} \quad (4.12)$$

is the Euler-Lagrange operator (see [H1], Section 6.2).

Specifically, if  $x$  and  $u$  are one-dimensional, and if

$$P = \int_{-\infty}^{+\infty} \rho(u, u_x, \dots, u_{x \dots x}^{(M)}, x, t) dx \quad (4.10')$$

then

$$\text{grad} P = \frac{\partial \rho}{\partial u} - D_x \left( \frac{\partial \rho}{\partial u_x} \right) + D_x^2 \left( \frac{\partial \rho}{\partial u_{xx}} \right) - \dots + (-1)^M D_x^M \left( \frac{\partial \rho}{\partial u_{x \dots x}^{(M)}} \right). \quad (4.13)$$

**Example 3.** Let

$$P = \int_{-\infty}^{+\infty} \left( -\frac{u_x^2}{2} + u^3 \right) dx. \quad (4.14)$$

Then  $P'[v] = \langle u_{xx} + 3u^2, v \rangle$ . Hence

$$\text{grad} P = u_{xx} + 3u^2. \quad (4.15)$$

If  $\gamma = u_{xx} + 3u^2$ , then  $\gamma' = D_x^2 + 6u = (\gamma')^+$ . Hence the condition for  $\gamma$  to be a gradient function is satisfied.

**Example 4.** Let

$$P = \int_{-\infty}^{+\infty} \left( \frac{1}{2} u H u_x + \frac{u^3}{3} \right) dx, \quad (4.16)$$

where  $H$  is the Hilbert transform 4.6. Here,  $P$  is not of the form 4.10, since the density  $\rho$  is not a differential function. Equation 4.8' yields:

$$\text{grad} P = H u_x + u^2. \quad (4.17)$$

## 4.2. Basic notions

Here we classify, as the basic notions, symmetries, conservation laws, recursion operators, and master symmetries. The notions, which refer to Hamiltonian structure, are treated in Section 4.3.

Here,  $\partial\sigma/\partial t$  denotes the partial derivative of  $\sigma$  with respect to its explicit dependence on  $t$ , and  $\llbracket\sigma, K\rrbracket$  is the commutator defined by

$$\llbracket\sigma(u, x, t), K(u, x, t)\rrbracket = \sigma'[K] - K'[\sigma] \quad (4.24)$$

$$\equiv \frac{\partial}{\partial \epsilon} [\sigma(u + \epsilon K(u, x, t), x, t) - K(u + \epsilon \sigma(u, x, t), x, t)]|_{\epsilon=0}.$$

In particular, if  $\sigma$  does not depend explicitly on  $t$ , Equation 4.23 has the form

$$\llbracket\sigma, K\rrbracket = 0. \quad (4.25)$$

If  $\sigma$  and  $K$  are differential functions, then the obvious relation

$$\sigma'[K] = \hat{X}_K(\sigma) \quad (4.26)$$

is valid and Equation 4.23 can be written in the form

$$\frac{\partial \hat{X}_\sigma}{\partial t} + [\hat{X}_K, \hat{X}_\sigma] = 0. \quad (4.27)$$

In Equation 4.27,  $\hat{X}_K$  and  $\hat{X}_\sigma$  are canonical Lie-Bäcklund operators 4.18. They generate the flows 4.1 and 4.19, respectively. Furthermore,  $[\hat{X}_K, \hat{X}_\sigma]$  is the usual Lie bracket of Lie-Bäcklund operators (see Chapter 1). Hence, the Lie bracket and the commutator 4.24 are connected by the equation

$$[\hat{X}_K, \hat{X}_\sigma] = \hat{X}_{\llbracket\sigma, K\rrbracket}. \quad (4.28)$$

It follows that, for differential functions,  $\llbracket\sigma, K\rrbracket$  coincides with the bracket  $\{\sigma, K\} = \sigma_* K - K_* \sigma$  of Ibragimov [1981], [1983]. Here, the notation 4.24 is taken for this bracket in order to distinguish it from the Poisson bracket for Hamiltonian systems.

Equation 4.23 is known as *the determining equation* for symmetries. Given  $K$ , the determining equation is a system of linear homogeneous first-order partial differential equations for  $\sigma$ :

$$\left(\frac{\partial}{\partial t} + \hat{X}_K - K'\right)[\sigma] = 0, \quad (4.23')$$

provided that  $K$  and  $\sigma$  are differential functions.

**Example 1.** For the Korteweg-de Vries equation 4.4, the function

$$\sigma = u_{xxxxx} + 10uu_{xxx} + 20u_xu_{xx} + 30u^2u_x \quad (4.29)$$

is a symmetry given by a differential function, and hence it is a Lie-Bäcklund symmetry (see, e.g., Ibragimov and Shabat [1979]).

**Example 2.** For the Benjamin-Ono Equation 4.5, the function

$$\sigma = (2u^3 + 3H(uu_x) + 3uHu_x - 2u_{xx})_x, \quad (4.30)$$

is a symmetry (Bock and Kruskal [1979]). It follows from the definition of the Hilbert transform 4.6 that  $\sigma$  is not a differential function. Hence, it is not a Lie-Bäcklund symmetry.

#### 4.2.2. Conservation laws

Let  $P: S \rightarrow \mathbb{R}$  be a functional.

**DEFINITION 4.2.** A functional  $P(u, x, t)$  is said to be conserved by the flow 4.1, if

$$D_t[P]|_{u_t=K} = 0,$$

or, equivalently, if

$$\frac{\partial P}{\partial t} + \langle \gamma, K \rangle = 0, \quad \gamma = \text{grad} P. \quad (4.31)$$

The vector  $\gamma \in S^*$  is called a conserved gradient for Equation 4.1.

Let the functional  $P$  be given by the integral 4.10. Then Equation 4.31 can be written in the differential form:

$$D_t[\rho]|_{u_t=K} + \text{Div}[V] = 0. \quad (4.32)$$

The differential function  $\rho$  is called a conserved density of order  $m$ , and  $V = \{V_i\}$  is known as a flux. Here,

$$\text{Div}[V] = \sum_{i=1}^d D_i[V_i].$$

Equation 4.31 is valid with  $\gamma$  defined by Equation 4.11:

$$\gamma = \frac{\delta P}{\delta u}. \quad (4.33)$$

Hence, the Euler-Lagrange operator maps a conserved density  $\rho$  into a conserved gradient  $\gamma$ .

The conserved gradient  $\gamma$  coincides with the characteristic of a conservation law (Olver [1986]), written in the following *characteristic form*:

$$D_t[\rho] + \text{Div}[V] = \gamma \cdot [u_t - K(u, x, t)]. \quad (4.34)$$

Thus,  $\gamma$  is an analog of an integrating factor well known for ordinary differential equations.

**DEFINITION 4.3.** A function  $\gamma: S \rightarrow S^*$  (it may depend explicitly on  $t, x$ ) is called a *conserved covariant* for Equation 4.1 if it satisfies the equation (see Fuchssteiner and Fokas [1981]):

$$\frac{\partial \gamma}{\partial t} + \gamma'[K] + (K')^+[\gamma] = 0. \quad (4.35)$$

**THEOREM 4.1.** Let the function  $\gamma$  be a gradient, i.e.,

$$(\gamma')^+ = \gamma'. \quad (4.36)$$

If  $\gamma = \text{grad} P$  is a covariant of Equation 4.1, then it is a conserved gradient of Equation 4.1, and the potential functional  $P$  is conserved by Equation 4.1.

If  $P$  is given by the integral 4.10 and if  $K$  in Equation 4.1 is a differential function, then Equation 4.35 is equivalent to the corresponding equations of Ibragimov [1983] (with one-dimensional  $x$  and  $u$ ):

$$D_t[\gamma]|_{u_t=K} + \sum_{j=0}^m (-D_x)^j \left[ \frac{\partial K}{\partial u_{x \dots x}^{(j)}} \cdot \gamma \right] = 0, \quad (4.35')$$

and of Olver [1986] (for  $(\gamma')^+ = \gamma'$ ):

$$\frac{\partial \gamma}{\partial t} + D_K^+[\gamma] + D_\gamma^+[K] = 0. \quad (4.35'')$$

The determining equation (4.35'') for integrals can be written in the form

$$\left(\frac{\partial}{\partial t} + \hat{X}_K - K'\right)^+[\gamma] = 0. \quad (4.35''')$$

**THEOREM 4.2.** (cf. Ibragimov [1983], Section 22.5). *The determining equations 4.23' and 4.35''' for symmetries and integrals, respectively, are mutually adjoint.*

**REMARK.** The function  $W(t) = \gamma(u, x, t)$  is a conserved gradient for Equation 4.1 iff it is a solution of the adjoint perturbation equation (cf. Equation 4.22):

$$W_t = -(K')^+(u, x, t)[W], \quad W \in S^*. \quad (4.37)$$

**Example 1.** (see Fuchssteiner and Fokas [1981]). For the KdV equation 4.4, the quantity

$$P = \int_{-\infty}^{+\infty} \left( \frac{u_{xx}^2}{2} + \frac{5}{2} u^2 u_{xx} + \frac{10}{4} u^4 \right) dx \quad (4.38)$$

is a conserved functional. The corresponding conserved gradient of  $P$  is

$$\gamma = \text{grad} P = u_{xxxx} + 10uu_{xx} + 5u_x^2 + 10u^3. \quad (4.39)$$

**Example 2.** (see Fuchssteiner [1983]). For the Benjamin-Ono equation 4.5, the functional

$$P = \int_{-\infty}^{+\infty} \left( -uu_{xx} - \frac{3}{2} u_x H u^2 + \frac{1}{2} u^4 \right) dx, \quad (4.40)$$

where  $H$  is Hilbert transform 4.6, is conserved by the flow 4.5 and the conserved gradient of  $P$  is

$$\gamma = \text{grad} P = 2u^3 + 3H(uu_x) + 3uHu_x - 2u_{xx}. \quad (4.41)$$

Note that the conserved density and gradient in Equations 4.38 and 4.39, respectively, are differential functions, while they are not differential functions for Equations 4.40 and 4.41.

### 4.2.3. Recursion operators

**DEFINITION 4.4.** A function  $R$  from  $S$  into the space of operators  $S \rightarrow S$  is called a recursion operator or a strong symmetry for Equation 4.1 if it maps

symmetries of Equation 4.1 into symmetries of the same equation (see Olver [1977], Ibragimov and Shabat [1979, 1980a], Fuchssteiner [1979]).

**THEOREM 4.3.** *A necessary and sufficient condition for  $R = R(u, x, t)$  to be a recursion operator of Equation 4.1 is that  $R$  commutes, when restricted on the solution manifold of Equation 4.23, with the operator of the determining equation 4.23 for symmetries. This condition is written:*

$$R_t + R'[K] - [K', R] = 0. \quad (4.42)$$

The zero in the right-hand side of Equation 4.42 means an operator, which annihilates any solution  $\sigma$  of Equation 4.23.

**Example.** For the KdV equation 4.4, a recursion operator due to A. Lenard (see Olver [1977], Ibragimov and Shabat [1979], Fuchssteiner and Fokas [1981]) is given by

$$R = D_x^2 + 4u + 2u_x D_x^{-1}. \quad (4.43)$$

Here and in what follows,  $D_x^{-1}$  is defined by

$$(D_x^{-1} f)(x) = \int_{-\infty}^x f(\xi) d\xi.$$

**THEOREM 4.4.** *Let  $R(u)$  be a recursion operator. Then the adjoint operator  $R^+(u)$  maps conserved covariants into conserved covariants.*

It follows that the operator  $R^+$  generates solutions  $\tilde{\gamma}$  of the determining equation 4.35 for integrals from a given solution  $\gamma$  of this equation:

$$\tilde{\gamma} = R^+(u)[\gamma].$$

However,  $\tilde{\gamma}$  may be not a gradient function because the constraint  $(\tilde{\gamma}')^+ = \tilde{\gamma}'$  may not be satisfied.

#### 4.2.4. Lie-Bäcklund and generalized symmetries

A recursion operator  $R$  can generate an infinite set of symmetries. Indeed, if  $\sigma$  is a symmetry and  $R$  is a recursion operator for Equation 4.1, then  $R^j[\sigma]$  ( $j = 1, 2, \dots$ ) are also symmetries of the same equation.

**DEFINITION 4.5.** A symmetry  $\sigma$  of Equation 4.1 is called a Lie-Bäcklund symmetry if  $\sigma$  is a differential function of an arbitrary finite order (see Chapter 1). We call  $\sigma$  a generalized symmetry if it is not a differential function.

Equation 4.30 gives an example of a generalized symmetry  $\sigma$ .

The following example provides first-order Lie-Bäcklund symmetries and recursion operators for semi-Hamiltonian systems.

**Example.** A first-order diagonal quasilinear system

$$u_t^i = v_i(u)u_x^i \quad (i = 1, 2, \dots, n; \quad n \geq 3) \quad (4.44)$$

is called semi-Hamiltonian (Tsarev [1985]) if  $v_i \neq v_j$  for  $i \neq j$  and if

$$[v_{i,u^j}/(v_j - v_i)]_{,u^k} = [v_{i,u^k}/(v_k - v_i)]_{,u^j} \quad (4.45)$$

for  $i \neq j \neq k \neq i$ . Here, the subscripts  $t, x, u^j$  denote the partial derivations with respect to these variables. For a semi-Hamiltonian system (4.44) a continuum set of first-order symmetries is generated by Lie equations:

$$u_t^i = w_i(u)u_x^i \quad (i = 1, 2, \dots, n), \quad (4.46)$$

where  $w_i(u)$  satisfies the linear system

$$w_{i,u^j} = \Gamma_{ij}^i(u)(w_j - w_i) \quad (j \neq i) \quad (4.47)$$

with  $\Gamma_{ij}^i = v_{i,u^j}/(v_j - v_i)$  ( $j \neq i$ ). Hence,

$$\sigma = \{w_i(u)u_x^i\}.$$

A recursion operator of the form

$$R = (AD_x + B)U_x^{-1}, \quad (4.48)$$

where  $A(u)$ ,  $B(u, u_x)$  are  $n \times n$  matrices, and

$$U_x = \begin{pmatrix} u_x^1 & & 0 \\ & \ddots & \\ 0 & & u_x^n \end{pmatrix},$$

is given in Teshukov [1989] for symmetries of Equation 4.44 under the following additional constraints:

$$S_{i,u^j} = \Gamma_{ij}^i (S_j - S_i) \quad (j \neq i) \quad (4.49)$$

where

$$S_i(u) = \sum_{k=1}^n c_k(u^k) \Gamma_{ik}^i(u) + d_i(u^i).$$

The recursion operator (4.48) generates the recursion formula

$$\bar{w}_i(u) = c_i(u^i) w_{i,u^i}(u) + d_i(u) w_i(u) + \sum_{k=1}^n c_k(u^k) \Gamma_{ik}^i(u) w_k(u). \quad (4.50)$$

Under the conditions 4.49,  $\bar{\sigma}_i = \bar{w}_i(u) u_x^i$  is a first-order symmetry of Equations 4.44 if  $\sigma_i = w_i(u) u_x^i$  is a symmetry.

In Sheftel' [1993], a second-order recursion operator is given in the form

$$R = (AD_x^2 + BD_x + C)U_x^{-1}, \quad (4.51)$$

where  $A, B, C$  are  $n \times n$  matrices. Existence conditions of this recursion have the same form 4.49:

$$B_{i,u^j} = \Gamma_{ij}^i (B_j - B_i) \quad (j \neq i) \quad (4.52)$$

However, here  $B_i(u)$  are different from  $S_i(u)$ . As a result, the constraints 4.52 are more liberal than Equation 4.49.

#### 4.2.5. Infinite symmetry algebras and conservation laws

Here we consider the one-dimensional  $x$  and use the notation of Chapter 1.

Let the right-hand side of Equation 4.1 be a differential function of order  $m$ :  $K = K(u, u_1, \dots, u_m)$ . Here  $K$  and  $u$  may be vectors. Then Equation 4.1 is a system of evolution equations.

**DEFINITION 4.6.** Equation 4.1 is said to be formally integrable if a solution of Equation 4.42 for a recursion operator  $R(u)$  exists and belongs to the class of formal operator power series

$$R^{(k)} = P_k(u) D_x^k + \dots + P_0(u) + P_{-1}(u) D_x^{-1} + P_{-2}(u) D_x^{-2} + \dots, \quad (4.53)$$

where the coefficients  $P_i(u)$  are differential functions. The integer  $k$  is termed the order of the operator  $R^{(k)}$ .

**THEOREM 4.5.** (see Ibragimov and Shabat [1980a] through [1980c], and Ibragimov [1983], Section 19.3). The set of solutions 4.53 for Equation 4.42 is closed under the multiplication and extraction of a root.

**COROLLARY.** If Equation 4.1 is formally integrable, then there exists a first-order solution of Equation 4.42:

$$L = R^{(1)} = r_1(u)D_x + r_0(u) + r_{-1}(u)D_x^{-1} + r_{-2}(u)D_x^{-2} + \dots \quad (4.54)$$

**DEFINITION 4.7.** The evolution equation 4.1 admits an infinite Lie-Bäcklund algebra if there exists a symmetry  $\sigma(u, u_1, \dots, u_N, x, t)$  of an arbitrarily high order  $N$ .

**THEOREM 4.6.** (Ibragimov and Shabat [1980a]-[1980c]). For any scalar equation 4.1, which admits an infinite Lie-Bäcklund algebra, Equation 4.42 for a recursion operator is solvable in the class of formal operator power series 4.53 of arbitrary order  $k$ .

**COROLLARY.** Under the conditions of Theorem 4.6, any solution of order  $m$  of Equation 4.42 has the form

$$R^{(m)} = K' + pD_x + q + rD_x^{-1} + \dots \quad (4.55)$$

For a motivation of the following definitions, see Mikhailov and Shabat [1985], [1986]; Mikhailov, Shabat, and Yamilov [1987]; Sokolov [1988]; and Shabat [1989].

**DEFINITION 4.8.** The coefficient  $P_{-1}$  in the formal power series 4.53 (or  $r_{-1}$  in the series 4.54) is termed the residue of the operator  $R$  (or of  $L$ ).

**DEFINITION 4.9.** The following differential functions of coefficients of the operator  $L = R^{(1)}$  form the canonical series of residues of the operator  $R = L^{(k)}$ :

$$\rho_{-1} = \frac{1}{r_1}, \quad \rho_0 = \frac{r_0}{r_1}, \quad \rho_i = \text{res}(L^i) \quad (i = 1, 2, \dots). \quad (4.56)$$

In other words, the canonical series of residues of the operator  $R^{(k)}$  is the set of

residues of its fractional powers:

$$\rho_i = \text{res}(R^{(k)\frac{i}{k}}), \quad i = -1, 1, 2, \dots$$

(see Gel'fand and Dikii [1976] and Manin [1979]), supplemented by the logarithmic residue  $\rho_0$ .

**THEOREM 4.7.** (Shabat and Sokolov [1982]). If Equation 4.1 possesses a conservation law of an arbitrarily high order, then Equation 4.42 for recursion operators is solvable in the class of the formal operator power series 4.53 of an arbitrary order  $k$ .

**THEOREM 4.8.** (Shabat [1989]). Let  $R$  be a recursion operator which satisfies Equation 4.42 and has the form 4.53. Then the traces of all canonical residues of a formal power series  $R$  are conserved densities for Equation 4.1:

$$D_t[\text{tr res} L] = D_x[V_0], \quad D_t[\text{tr res} L] = D_x[V_1],$$

$$D_t[\text{tr res} L^i] = D_x[V_i], \quad i = -1, 1, 2, \dots, \quad (4.57)$$

where  $D_t$  is calculated according to Equation 4.1 and the fluxes  $V_i$  are differential functions.

#### 4.2.6. Hereditary recursion operators

**DEFINITION 4.10.** (Fuchssteiner [1979]). A recursion operator is said to be hereditary (or Nijenhuis) if it generates an Abelian symmetry algebra out of commuting symmetries.

Such recursion operators are also called Kähler (Magri [1978]) or regular (Gel'fand and Dorfman [1979, 1980]).

More specifically, let the flow  $u_t = v$  commute with the flows  $u_t = w$ ,  $u_t = Rw$  and let the flow  $u_t = w$  commute with the flow  $u_t = Rv$ , where  $v, w$  are arbitrary smooth functions. Then a recursion operator  $R$  is hereditary iff we require that the flows  $u_t = Rv$  and  $u_t = Rw$  also commute.

Assume for simplicity that  $\partial R / \partial t = 0$ , i.e.,  $R = R(u): S \rightarrow S$  for  $u \in S$ .

**THEOREM 4.9.** (Fuchssteiner [1979]). A recursion operator  $R(u)$  is hereditary iff the commutator  $[R'(u), R(u)]$  is a symmetric bilinear operator for all  $u \in S$ .

**THEOREM 4.14.** A function  $\tau \in L^*$  is a master symmetry of Equation 4.1 of degree  $n$  iff, for any symmetries  $\sigma_1, \sigma_2, \dots, \sigma_n$  of Equation 4.1 which belong to  $L$ , the function  $\hat{\sigma}_1 \cdot \hat{\sigma}_2 \cdot \dots \cdot \hat{\sigma}_n \tau$  is also a symmetry of Equation 4.1, which belongs to  $L$ .

It is clear that all master symmetries are  $K$ -generators (of the same degree), but the inverse is not true without additional conditions.

**THEOREM 4.15.** Let  $K^\perp$  be abelian. Let  $\tau$  be a  $K$ -generator of degree  $n$  such that  $\hat{\sigma}_1 \cdot \hat{\sigma}_2 \cdot \dots \cdot \hat{\sigma}_n \tau \in L$  for arbitrary  $\sigma_1, \dots, \sigma_n \in K^\perp$ . Then  $\tau$  is a master symmetry of degree  $n$ .

**Example.** For Equation 4.5 with  $K(u) = Hu_{xx} + 2uu_x$ , where  $H$  is the Hilbert transform 4.6, the function

$$\tau_1 = -6xK(u) - 6u^2 - 9Hu_x \quad (4.64)$$

is a master symmetry of degree 1. It generates an infinite sequence of commuting symmetries of Equation 4.5 by means of the relations

$$K_1 = K, \quad K_2 = \frac{1}{6} \llbracket \tau_1, K \rrbracket, \dots, K_{n+1} = \frac{1}{6} \llbracket \tau_1, K_n \rrbracket. \quad (4.65)$$

Infinitely many master symmetries of degree 1 are obtained by

$$\tau_n = \llbracket x, K_{n+1} \rrbracket. \quad (4.66)$$

They are related through Equation 4.62' to time-dependent symmetries of Equation 4.5 linear in  $t$ :

$$\sigma_{\tau_n}(t) = \tau_n + t \llbracket K_1, \tau_n \rrbracket. \quad (4.67)$$

Master symmetries  $\tau_{m,n}$  of degree  $m$  are given by the recursion ( $\tau_{1,n} = \tau_n$ ):

$$\tau_{2,n} = \llbracket x, \tau_n \rrbracket, \dots, \tau_{m+1,n} = \llbracket x, \tau_{m,n} \rrbracket \quad (m = 1, 2, \dots; n = 1, 2, \dots). \quad (4.68)$$

They are related through Equation 4.62' to time-dependent symmetries of Equation 4.5 which are polynomial of degree  $m$  in  $t$ :

$$\sigma_{\tau_{m,n}}(t) = \sum_{i=0}^m \frac{t^i}{i!} (\hat{K})^i \tau_{m,n}. \quad (4.69)$$

### 4.3. Recursion operators in Hamiltonian formalism

The major part of equations, which have infinitely many symmetries, possesses an additional structure which is a continual analog of Hamiltonian formalism of classical mechanics (see Manin [1979]). We proceed now to expose the related concepts.

#### 4.3.1. Noether and inverse Noether operators

**DEFINITION 4.13.** (see Fuchssteiner and Fokas [1981]). Let  $\theta$  be a function from  $S$  into the space of operators  $S^* \rightarrow S$ , which may explicitly depend on  $x, t$ , i.e.,  $\theta = \theta(u, x, t)$  — for any  $u \in S$ . The function  $\theta$  is said to be Noether operator for Equation 4.1 if, for any  $u \in S$  satisfying Equation 4.1,  $\theta(u, x, t)$  maps any conserved covariant  $\gamma$  of Equation 4.1 into a symmetry  $\sigma$  of Equation 4.1:

$$\sigma = \theta \gamma \quad (4.70)$$

**THEOREM 4.16.** (see Oevel and Fokas [1982]).  $\theta(u, x, t)$  is a Noether operator for Equation 4.1 iff for any  $u \in S$  it satisfies the equation:

$$\frac{\partial \theta}{\partial t} + \theta'[K] - \theta(K')^+ - K'\theta = 0. \quad (4.71)$$

**DEFINITION 4.14.** Let  $J$  be a function from  $S$  into the space of operators  $S \rightarrow S^*$ :  $J = J(u, x, t)$  for any  $u \in S$ . The function  $J$  is said to be an inverse Noether operator for Equation 4.1 iff, for any  $u \in S$  satisfying Equation 4.1,  $J(u, x, t)$  maps any symmetry  $\sigma$  of Equation 4.1 into a conserved covariant  $\gamma$  of Equation 4.1:

$$\gamma = J\sigma. \quad (4.72)$$

**THEOREM 4.17.** (see Oevel and Fokas [1982]).  $J(u, x, t)$  is an inverse Noether operator for Equation 4.1 iff for any  $u \in S$  it satisfies the equation:

$$\frac{\partial J}{\partial t} + J'[K] + JK' + (K')^+J = 0. \quad (4.73)$$

If  $\theta$  satisfies Equation 4.71 and  $\theta^{-1}$  exists, then  $J = \theta^{-1}$  satisfies Equation 4.73.

for arbitrary  $a, b, c \in S^*$ .

To account for the case when the operator  $\theta$  is not invertible we accept the following definition.

**DEFINITION 4.16.** *An operator-valued function  $\theta(u, x, t): S^* \rightarrow S$ ,  $u \in S$ , which is skew-symmetric  $\theta^+ = -\theta$ , is called implectic (inverse-symplectic) or a Hamiltonian operator if the bracket defined by the formula 4.78 satisfies the Jacobi identity:*

$$\langle b, \theta'[\theta a]c \rangle + \langle c, \theta'[\theta b]a \rangle + \langle a, \theta'[\theta c]b \rangle = 0 \quad (4.79)$$

for all  $a, b, c \in S^*$ .

Here the existence of  $\theta^{-1}$  is not assumed. The left-hand side of Equation 4.79 is called a Schouten bracket (see Lichnerowicz [1977], Olver [1984], Kosmann-Schwarzbach [1986], Kosmann-Schwarzbach and Magri [1988, 1989]).

Consider obvious examples of Hamiltonian operators.

1. A constant skew-symmetric operator.
2. The inverse of a symplectic operator, if it exists.

**THEOREM 4.20.** *Let  $\theta(u)$ ,  $u \in S$  be skew-symmetric operators  $S^* \rightarrow S$ . Then the following assertions are equivalent:*

1.  $\theta$  is Hamiltonian;
2.  $\theta$  is a Noether operator for every evolution equation of the form:

$$u_t = \theta(u)h(u) \quad (4.80)$$

where  $h(u)$  is a gradient function:  $h(u) = \text{grad}H(u)$ ;

3. for all gradient functions  $f$  and  $g$  we have:

$$(\theta f)'[\theta g] - (\theta g)'[\theta f] = \theta(f, \theta g)'[\cdot]. \quad (4.81)$$

**Example.**  $\theta_1 = D_x$  and  $\theta_2$  in Equation 4.74 are Hamiltonian operators.  $J_1 = D_x^{-1}$  is a symplectic operator.

### 4.3.3. Hamiltonian systems

The following concepts and results are due to Magri [1978], Gel'fand and Dorfman [1979, 1980], Fuchssteiner and Fokas [1981], Oevel and Fokas [1982] (see also Olver [1980, 1986]).

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